# TBP MATH33A Review Sheet

November 24, 2018

**General Transformation Matrices:**

| Function | Implementation |
|---|---|
| **Scaling by *k*** | If we want to scale $I_2$ by $k$, we use the following:<br>$k * I_2 = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ |
| **Orthogonal projection onto line *L*** | Given a unit vector $\vec{u}$ that is parallel to $L$ such that<br>$\vec{\boldsymbol{u}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$<br>The matrix<br>$\begin{bmatrix} {u_1}^2 & u_1 u_2 \\ u_1 u_2 & {u_2}^2 \end{bmatrix}$<br>generates an orthogonal projection onto line $L$where ${u_1}^2 + {u_2}^2 = 1$. |
| **Reflection about a line** | In order to reflect about a line, we utilize a matrix of the form<br>$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$<br>where $a^2 + b^2 = 1$. |
| **Rotation through angle $\theta$** | Rotation through an angle $\theta$ involves a matrix of the form<br>$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$<br>or, more generally,<br>$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$<br>where $a^2 + b^2 = 1$. |
| **Rotation through angle $\theta$ combined with scaling by *r*** | This matrix takes the form<br>$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$<br>which is equivalent to<br>$r \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ |
| **Horizontal shear** | Matrices of the form<br>$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$<br>where $k$ is an arbitrary constant. |

| | |
|---|---|
| **Vertical shear** | Matrices of the form $$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$ where $k$ is an arbitrary constant. |

**Chapter 1**

- **Rank** shows dimension of im($A$) where $A$ is an arbitrary matrix
- If rank($\tilde{A}$) = rank($A$), where $\tilde{A}$ is the augmented matrix of $A$, then the system is solvable and the dimension of the solution space is equal to the number of free variables.
  - Furthermore, if, for the $m$x$n$ matrix, $m < n$ and the system is solvable, there are *infinitely many solutions*
- A transformation $T$ is linear iff
  - $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^m$
  - $T(k\vec{v}) = kT(\vec{v})$ for all vectors $\vec{v}$ in $\mathbb{R}^m$ and all scalars $k$

**Chapter 2**

- Earlier, we defined an orthogonal projection onto line $L$ where the unit vector $\vec{u}$ is on the line. We now define the more general case with a vector $\vec{u}$ that is on the line but not a unit vector. In this case
$$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$
and the matrix is
$$\frac{1}{{w_1}^2 + {w_2}^2} \begin{bmatrix} {w_1}^2 & w_1 w_2 \\ w_1 w_2 & {w_2}^2 \end{bmatrix}$$
- Furthermore, a reflection with respect to line $L$ with an arbirtrary vector $w$ on $L$ as defined before can be found using
$$\frac{1}{{w_1}^2 + {w_2}^2} \begin{bmatrix} {w_1}^2 - {w_2}^2 & 2w_1 w_2 \\ 2w_1 w_2 & {w_2}^2 - {w_1}^2 \end{bmatrix}$$
which more generally takes the form
$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$
where, as before, $a^2 + b^2 = 1$.
- Generally, for subspace $L$ and $V$ s.t. $V = L^\perp$,
$$Ref_V(\vec{x}) = -Ref_L(\vec{x})$$

- The projection matrix onto a space where $\vec{u}$ is a unit vector in the space of interest ()and defined in the same way as before) is found by computing the matrix

$$\begin{bmatrix} u_1\vec{\boldsymbol{u}} & u_2\vec{\boldsymbol{u}} \end{bmatrix}$$

for the 2D case and

$$\begin{bmatrix} u_1\vec{\boldsymbol{u}} & u_2\vec{\boldsymbol{u}} & u_3\vec{\boldsymbol{u}} \end{bmatrix}$$

- Finally, we define that the matrix for reflection about a subspace $V$ is equal to the identity matrix of that dimension minus the projection matrix.

$$A_R V = I_d - A_P V$$

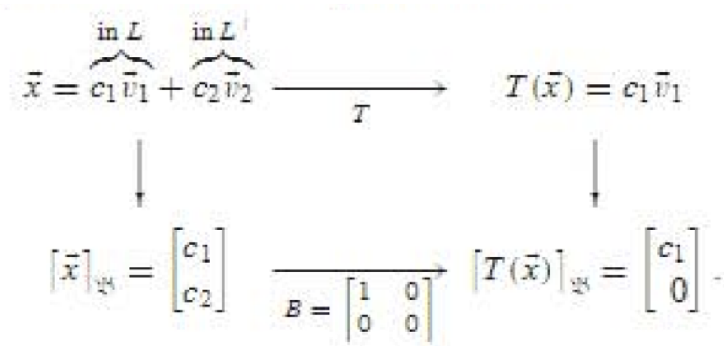
**Chapter 3**

- We begin by defining a subspace as something that is closed under addition and allows for scalar multiplication.
- Furthermore, if $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ is a linear transformation from $\mathbb{R}^{\mathrm{m}}$ to $\mathbb{R}^{\mathrm{n}}$, then
  - $ker(T) = ker(A)$ is a subspace of $\mathbb{R}^{\mathrm{m}}$
  - $im(T) = im(A)$ is a subspace of $\mathbb{R}^{\mathrm{n}}$
- We define a basis to be a set of vectors in substance $V$ of $\mathbb{R}_{\mathrm{n}}$ s.t. $B = (\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3, ...\vec{\boldsymbol{v}}_{\mathrm{m}})$
- Any vector $\vec{\boldsymbol{x}}$ in $V$ can be written as $\vec{\boldsymbol{x}} = c_1\vec{\boldsymbol{v}}^1 + c_1\vec{\boldsymbol{v}}^2 + c_1\vec{\boldsymbol{v}}^3 + {}_1\vec{\boldsymbol{v}}^{\mathrm{m}}$ where $c_1, c_2, c_3, ...c_m$ are the $\beta$ coordinates of $\vec{\boldsymbol{x}}$ and the vector

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ . \\ . \\ . \\ c_m \end{bmatrix}$$

is defined as the $\beta$ coordinate vector of $\vec{\boldsymbol{x}} = [\vec{\boldsymbol{x}}]_\beta$

- We see $\vec{\boldsymbol{x}} = B\ [\vec{\boldsymbol{x}}]_\beta$
- To get a better idea of the relationship between transformations and bases, we refer to the following diagram from page 152 of the Bretscher textbook.




- To explain this diagram further, we start at the top left corner and consider an arbitrary vector $\vec{x}$ with a basis as described above that is on an arbitrary line $L$. In order to describe the vector on the line, we need to use two basis vectors and two constants. Looking at the bottom left of the diagram, we see that the coordinate vector of $\vec{x}$ is defined with two different coefficients.
- However, let us consider now $\vec{x}$ the basis so that one of the basis vectors $\vec{v}_1$ is on $L$. This reduces the coordinate vector so that only the coefficient $c_1$ is relevant. The equivalent transformation for this change in basis is shown so that $T$ transforms

  $$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

  into

  $$\begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

- We observe a parallelism in the transformation matrix and the coordinate vector.

**Chapter 5**

- We define a set of orthonormal vectors to be a set of all unit vectors that are all perpendicular to each other and are linearly independent.
- We also define a transform of $A$ with notation $A^{\mathrm{T}}$ such that, if

  $$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

  then

  $$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

- Consequently, $\vec{v}^T = v \cdot w$.
- The orthogonal projection of $\vec{v}^T$ onto $V$ is denoted as $\vec{v}^{\|}$

- if $V$ is a subspace of $IR^{\text{n}}$ with orthonormal basis $\vec{\boldsymbol{u}}_1,\ \vec{\boldsymbol{u}}_2,\ \vec{\boldsymbol{u}}_3,\ ....\ \vec{\boldsymbol{u}}_m$

$$proj_V\vec{\boldsymbol{x}} = \vec{\boldsymbol{v}}^{\|} = (\vec{\boldsymbol{u}}_1 \cdot \vec{\boldsymbol{x}})\vec{\boldsymbol{u}}_1 + .... + (\vec{\boldsymbol{u}}_m \cdot \vec{\boldsymbol{x}})\vec{\boldsymbol{u}}_m$$

- Cauchy-Schwartz inequality: $|\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}}| \leq ||\vec{\boldsymbol{x}}|| * ||\vec{\boldsymbol{y}}||$
- Gram-Schmidt process: convert an old base to a new base
  - First, we consider a basis with vectors

    $$\vec{\boldsymbol{v}}_1,\ \vec{\boldsymbol{v}}_2,\ ....\vec{\boldsymbol{v}}_m$$

    and a new basis with vectors

    $$\vec{\boldsymbol{u}}_1,\ \vec{\boldsymbol{u}}_2,\ ...\ \vec{\boldsymbol{u}}_m$$

    found by

    $$\vec{\boldsymbol{u}}_1 = \tfrac{1}{||\vec{\boldsymbol{v}}_1||}\ \vec{\boldsymbol{v}}_1,\ \vec{\boldsymbol{u}}_2 = \tfrac{1}{||\vec{\boldsymbol{v}}_2||}\ \vec{\boldsymbol{v}}_2^{\perp},\ ...\ \vec{\boldsymbol{u}}_m = \tfrac{1}{||\vec{\boldsymbol{v}}_m||}\ \vec{\boldsymbol{v}}_m^{\perp}$$

    where we define

    $$\vec{\boldsymbol{v}}_j^{\perp} = \vec{\boldsymbol{v}}_j \text{ - } (\vec{\boldsymbol{u}}_1 \cdot \vec{\boldsymbol{x}}_j)\vec{\boldsymbol{u}}_1 \text{ - ... - } (\vec{\boldsymbol{u}}_{\text{ j - 1}} \cdot \vec{\boldsymbol{x}}_j)\vec{\boldsymbol{u}}_{\text{ j - 1}}$$

- Q-R factorization is a method we use to convert a matrix $R$ from the old base to a new base.
  - We use the equation $M = QR$ where M is a matrix with the basis vectors

    $$\vec{\boldsymbol{v}}_1,\ \vec{\boldsymbol{v}}_2,\ ....\vec{\boldsymbol{v}}_m$$

    as its columns and Q is a matrix with the unit basis vectors

    $$\vec{\boldsymbol{u}}_1,\ \vec{\boldsymbol{u}}_2,\ ...\ \vec{\boldsymbol{u}}_m$$

    as its columns.
  - This equation can also be rewritten as $R = Q^T A$
  - Note that, in using this, $Q$ must have orthonormal columns and $R$ is an upper triangular matrix with positive diagonals.
- A transformation is classified as an orthogonal transformation if it preserves length
- If $T(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ is an orthogonal transformation, $A$ is an orthogonal matrix.
- In general, an orthogonal matrix is invertable, its determinant is not equal to zero, the matrix multiplied by its transpose is the identity matrix, and the inverse of the matrix is equal to its transpose.
- The matrix of an orthogonal projection is $P = QQ^T$ where

  $$Q = \begin{bmatrix} \vec{\boldsymbol{u}}_1 & \vec{\boldsymbol{u}}_2 & ... & \vec{\boldsymbol{u}}_m \end{bmatrix}$$

- General properties of transpose:
  - $(A + B)^T = A^T + B^T$
  - $(kA)^T = kA^T$

- $(AB)^T = B^T A^T$
- $rank(A^T) = rank(A)$
- $(A^T)^{-1} = (A^{-1})^T$
- $(Im(A))^\perp = ker(A^T)$

- Least square solution
  - $(A^T A)\vec{\boldsymbol{x}}^\star = A^T \vec{\boldsymbol{b}}$
  - If $A^T A$ is invertible/linearly ind, then $\vec{\boldsymbol{x}}^\star = (A^T A)^{-1} A^T \vec{\boldsymbol{b}}$ and is unique
  - For $A\vec{\boldsymbol{x}}^\star = b$, $A\vec{\boldsymbol{x}}^\star$ is the orthogonal projection of $\vec{\boldsymbol{b}}$ onto the image of $A$.

- The matrix of an orthogonal projection is

$$A(A^T A)^{-1} A^T \vec{\boldsymbol{b}}$$

**Chapter 6**

- The determinant of an (upper or lower) triangular matrix is the product of the diagonal entries of the matrix.

- $det(A^T) = det(A)$

- Determinant operations:
  - Swapping rows results in $x * (-1)$ where $x$ is the determinant
  - Dividing a row by scalar $k$ results in $x * (\frac{1}{k}) * det(A)$
  - A square matrix is invertivle if and only if $det(A) = 0$
  - To make it easier to solve for the determinant of a given matrix $A$, we turn $A$ into some upper triangular matrix $B$ and find that

$$det(A) = (-1)^s * k_1 * k_2 * .... * k_r * det(B)$$

    where $s$ is the number of row swaps, $k$ is a scalar that we divide a row in $A$ by, and $r$ is the number of scalars we the rows of $A$ by.
  - $det(AB) = (det(A)) * (det(B))$
  - $det(A^m) = (det(A))^m$
  - $det(A^{-1}) = \frac{1}{det(A)} = (det(A))^{-1}$

**Chapter 7**

- An eigenvector is a vector $\vec{\boldsymbol{v}}$ such that

$$A\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}}$$

  where $\lambda$ is an eigenvalue

- An eigenbasis is a set of eigenvectors

$$\begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} & ... & \vec{v_n} \end{bmatrix}$$

- $B = S^{-1}AS$ or $A = SBS^{-1}$ where

$$S = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} & ... & \vec{v_n} \end{bmatrix}$$

$$B = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ ... & ... & ... & ... \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

- The trace of a matrix is defined as

$$tr(A) = a_{11} + a_{22} + ... + a_{nn}$$

  where $a_{ii}$ is the value of matrix at row $i$ and column $i$.

- The roots of $f_A(\lambda) = det(A - \lambda I)$.
- An eigenspace is defined as

$$E_\lambda = \ker(A - \lambda I_n)$$
or
$$\{\vec{v} \text{ in } \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\}$$

- Consequently, $dim(E_\lambda)$ = geometric multiplicity.

**Complex Numbers**

- z = a + b$i$ = $r(cos\theta + sint\theta i)$
- $e^{i\theta} = cos\theta + sin\theta i$
- Thus, z = r$e^{i\theta}$
- $\bar{z}$= a - b$i$
- $\bar{z}$ * z = $a^2$ + $b_2$ = $|z|^2$ = $r^2$
- If $A$ has eigenvalues a ± $i$b where b ≠ 0 and $\vec{v} + i\vec{w}$ is an eigenvector of $A$ with eigenvalue a ± $i$b, then $S^{-1}AS$ =

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

  where

$$S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}$$

- Alternatively,
$$A = S \begin{bmatrix} a & b \\ -b & a \end{bmatrix} S^{-1}$$
where
$$S = \begin{bmatrix} \vec{\boldsymbol{v_1}} & \vec{\boldsymbol{v_2}} \end{bmatrix}$$
and $\vec{\boldsymbol{v_1}} + \vec{\boldsymbol{v_2}}i$ is an eigenvector for a $\pm$ $i$b.

**Chapter 8**

- A symmetric matrix is defined as $A = A^T$.
- If $A_{\text{nn}}$ is symmetric matrix,
  - There exists an orthonormal eigenbases (and thus the matrix is orthogonally diagonalizable).
  - All eigenvalues are real ($\bar{\lambda} = \lambda$), there are $n$ real eigenvalues, and an algebraic multiplicity of $n$.
  - $x^T(Ty) = y^T(Tx)$
  - For any orthonormal basis $\beta$, $[T]_\beta$ is symmetric.
  - If $\vec{\boldsymbol{v_1}}$ and $\vec{\boldsymbol{v_2}}$ are eigenvectors, of $A$ with distinct eigenvalues $\lambda_1$ and $\lambda_2$, then $\vec{\boldsymbol{v_1}} * \vec{\boldsymbol{v_2}} = 0$ and $\vec{\boldsymbol{v_1}}$ is orthogonal to $\vec{\boldsymbol{v_2}}$.
- To perform an orthogonal diagonalization of a symmetric matrix $A$,
  - Find the eigenvalues of $A$, and find a basis of each eigenspace.
  - Use the Gram-Schmidt process, find an orthonormal basis of each eigenspace.
  - Forma n orthonormal eigenbases $\vec{\boldsymbol{v_1}}, \vec{\boldsymbol{v_2}}, \vec{\boldsymbol{v_3}}, ..., \vec{\boldsymbol{v_n}}$ for $A$ by concatenating the orthonormal basis you found in the second part such that
  $$S = \begin{bmatrix} \vec{\boldsymbol{v_1}} & \vec{\boldsymbol{v_2}} & \vec{\boldsymbol{v_3}} & ... & \vec{\boldsymbol{v_n}} \end{bmatrix}$$
  From this, $S$ is orthogonal and $S^{-1}AS$ will be diagonal.
- Quadratic forms:
  - $q(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}} * A\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^T A\vec{\boldsymbol{x}} = \lambda_1 c_1^2 + \lambda_2 c_2^2 + ... + \lambda_n c_n^2$
    where we define $A$ to be a symmetric $n$x$n$ matrix, $\beta$ to be an orthonormal eigenbasis for $A$ with associated $\lambda_1 c_1^2, \lambda_2 c_2^2, ..., \lambda_n c_n^2$, and $c_i$ to be the coordinates of $\vec{\boldsymbol{x}}$ with respect to $\beta$.
- Orthonormal coordinate transformation
$$x = Sy \text{ to } y = S^{-1}x = S^T x$$
such that
$$Q(\vec{\boldsymbol{y_1}}, \vec{\boldsymbol{y_2}}, ...\vec{\boldsymbol{y_n}}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2$$

- $A$ is
  - positive definite if q($\vec{\boldsymbol{x}}$) is positive for all nonzero $\vec{\boldsymbol{x}}$ in $\mathbb{R}^{\mathrm{n}}$ and iff $det(A)^{(m)}$ ¿ 0 for all $m = 1, 2, ..., n$. This implies that all eigenvalues are positive.
  - positive semidefinite if q($\vec{\boldsymbol{x}}$) $\geq 0$ for all $\vec{\boldsymbol{x}}$ in $\mathbb{R}^{\mathrm{n}}$. This implies that all eigenvalues are positive or zero.
  - indefinite if q($\vec{\boldsymbol{x}}$) takes positive as well as negative values.
- The principal axes for $q(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}} * A\vec{\boldsymbol{x}}$ with $A$ being a symmetric $n$x$n$ matrix with $n$ distinct eigenvalues are the eigenspaces of $A$.
- Conside the curve defined by $q(\vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = a\vec{\boldsymbol{x}}_1^2 + b\overrightarrow{\boldsymbol{x_1}}\overrightarrow{\boldsymbol{x_2}} + c\overrightarrow{\boldsymbol{x_2}}^2 = 1$ with eigenvalues $\lambda_1$, $\lambda_2$ of matrix
$$\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$$
of $q$. If both $\lambda_1$, $\lambda_2$ are positive, the curve is an ellipse, if one is positive and one is negative, the curve is a hyperbola.
- A linear transformation is described as $L(\vec{\boldsymbol{x}}) = A\vec{\boldsymbol{x}}$ and is an invertivle linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ where $A^TA$ is symmetric, the image of the unit circle under $L$ is an ellipse $E$, and the lengths of semimajor and semiminor axes of $E$ are $\sigma_1$ and $\sigma_2$ of $A$.
- We define the following equation:
$$A = U\Sigma V^T$$
where
  - $V$ is an orthonormal eigenbasis for $A^TA$ with respect to eigenvalues defined such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r \geq 0 = \lambda_{\mathrm{r}\,+\,1} \geq ... \geq 0 = \lambda_m$ with corresponding singular values (defined such athat singular values are the square root of $\lambda$) $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r \geq 0$.... We associate the columns of $V$ to be the orthonormal vectors $\overrightarrow{\boldsymbol{v_1}}, \overrightarrow{\boldsymbol{v_2}}, \overrightarrow{\boldsymbol{v_3}}, ...\overrightarrow{\boldsymbol{v_n}}$ in $\mathbb{R}^{\mathrm{n}}$ such that $L(\overrightarrow{\boldsymbol{v_1}}), L(\overrightarrow{\boldsymbol{v_2}}), L(\overrightarrow{\boldsymbol{v_3}}), ...L(\overrightarrow{\boldsymbol{v_n}})$ are orthogonal.
  - $\Sigma =$
$$\begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_r & 0 & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
  - $U =$
$$\begin{bmatrix} \overrightarrow{\boldsymbol{u_1}} & ... & \overrightarrow{\boldsymbol{u_2}} & \vec{\boldsymbol{u}}_{\mathrm{r}\,+\,1} = 0 & 0 & 0 \end{bmatrix}$$
where $\overrightarrow{\boldsymbol{u_i}}$ are ybut vectors defined by $\overrightarrow{\boldsymbol{u_1}} = \frac{1}{\sigma_1}A\overrightarrow{\boldsymbol{v_1}}$ .... $\overrightarrow{\boldsymbol{u_r}} = \frac{1}{\sigma_r}A\overrightarrow{\boldsymbol{v_r}}$ such that $A\overrightarrow{\boldsymbol{v_i}} = \sigma_1\overrightarrow{\boldsymbol{u_i}}$ for $i = 1, 2, 3....r$ and $A\overrightarrow{\boldsymbol{v_i}} = \bar{0}$ for $i = r + 1, ...m$.
  - Furthermore, if $A$ is an $n$ x $m$ matrix of rank $r$, then singular values $\sigma_1$, $\sigma_2$, ... $\sigma_r$ are nonzero while $\sigma_{\mathrm{r}\,+\,1}$, ..... $\sigma_m$ are zero.