Math 32B Review Sheet

Tau Beta Pi - Boelter 6266

Contents

1 Double Integrals

The basic premise of calculating a double integral is calculating the volume underneath a surface, just as the basic premise of calculating a single integral is finding the area under the line of a function. An example below shows a parabaloid where the function is $z = x^2 + y^2$.

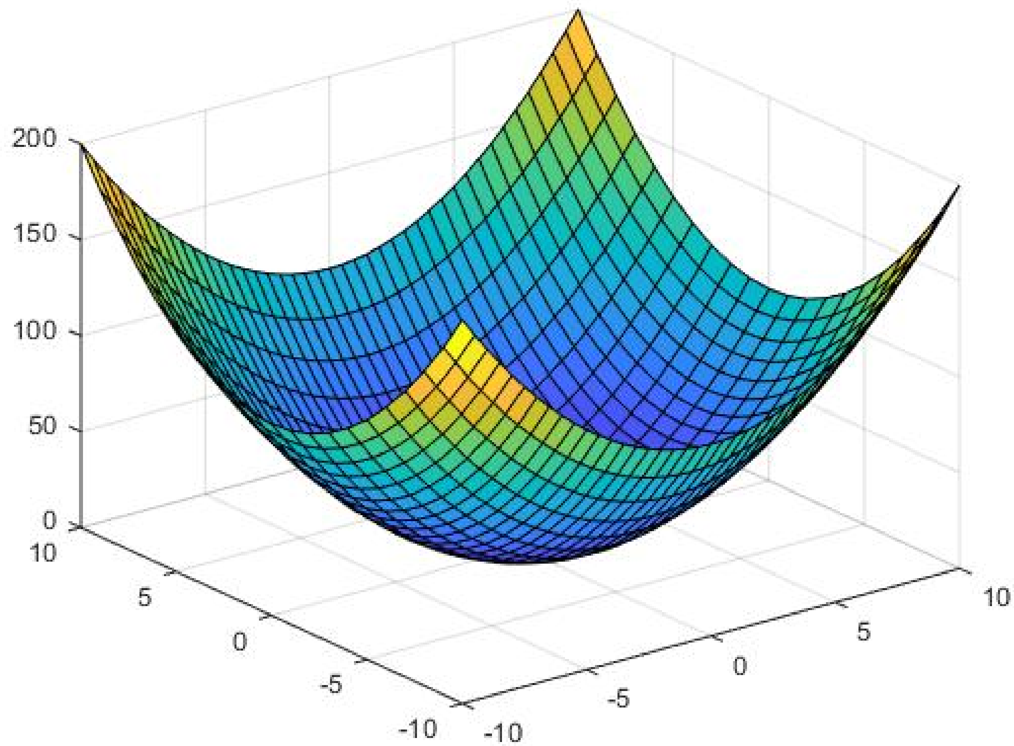

Figure 1: The graph of the surface $z = x^2 + y^2$ is displayed. (Graph produced via Matlab)

We can see that many of the rules of calculating areas under curves also translate to calculate volumes under surfaces. For example, if we call the domain $(-10, 10) \times (-10, 10)$ as D (essentially x can take values from -10 to 10 and y can take values from -10 to 10 as shown in Figure 1), then the area under the paraboliod shown in Figure 1 is

$$\iint_D x^2 + y^2 \, \mathrm{d}A$$

In general, calculating the (possibly signed) volume under the function $z = f(x, y)$ is

$$\iint_D f(x, y) \, \mathrm{d}A$$

As you might imagine, to calculate the volume "over" the paraboloid instead, for this image, the volume between $x^2 + y^2$ and 200 is

$$\iint_D 200 - (x^2 + y^2) \, \mathrm{d}A$$

Double integrals seem to be fairly easy to visualize, so the hardest problem is usually computation, which has more nuances than in the single variable case. It is somewhat hard to show graphically, but the idea behind computing the double integral over a rectangle is to treat dA as the product dxdy (or vice versa; more on that later). Then, the integral becomes

$$\iint_D x^2 + y^2 \,\mathrm{d}A = \int_{-10}^{10}\int_{-10}^{10} x^2 + y^2 \,\mathrm{d}x\mathrm{d}y$$

Holding y constant while integrating along x, we obtain

$$\iint_D x^2 + y^2 \,\mathrm{d}A = \int_{-10}^{10}\int_{-10}^{10} x^2 + y^2 \,\mathrm{d}x\mathrm{d}y = \int_{-10}^{10} \frac{1}{3}x^3 + xy^2 \,]_{-10}^{10} \,\mathrm{d}y \;= \int_{-10}^{10} \frac{2000}{3} + 20y^2 \,\mathrm{d}y$$

This last integral can be evaluated as a normal single integral from single variable calculus.

$$\iint_D x^2 + y^2 \,\mathrm{d}A = \int_{-10}^{10} \frac{2000}{3} + 20y^2 \,\mathrm{d}y \;= \frac{2000}{3}y + \frac{20}{3}y^3]_{-10}^{10} = \frac{80000}{3}$$

This might seem like an abnormally large number, but for reference, the box with the same base domain and height 100 has volume

$$20 \times 20 \times 100 = 40000 = \frac{12000}{3}$$

so this number is a bit easier to swallow.

In general, we can compute all volumes over rectangles in this fashion, calculating

$$\int_c^d \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y$$

where $\{x, y\} \in (a,b) \times (c,d)$ is the domain of integration. This idea is the same as starting out with the mixed derivative

$$\frac{\partial^2 f}{\partial x \partial y}$$

and "finding f". Going forward, we have

$$f \xrightarrow[\frac{\partial}{\partial y}]{1} \frac{\partial f}{\partial x} \xrightarrow[\frac{\partial}{\partial x}]{2} \frac{\partial^2 f}{\partial x \partial y}$$

In the second step, we differentiate with respect to x while holding y constant, so to invert that step, we need to integrate x while holding y constant. Similarly, in the first step, we differentiate y with x constant, so to get back the value of f, we want to integrate y while holding x constant. Since we have equality of mixed derivatives, integrating the other way makes no difference in either case.

Another way to think about it is to consider the inner integral as the area of a cross section and to integrate that area over the range of cross sections. For example, when integrating

$$\int_c^d \int_a^b f(x,y)\mathrm{d}x\mathrm{d}y$$

we can first let $S(y) = \int_a^b f(x,y)\mathrm{d}x$. S is only a function of y because we are integrating out x. For constant y, this corresponds to the area of a cross section under the surface. Then, we can integrate with respect to y, which gives

$$\int_c^d \int_a^b f(x,y)\mathrm{d}x\mathrm{d}y = \int_c^d S(y)\mathrm{d}y$$

which allows us to calculate the double integral as the familiar single integral we all know. Integrating the area of each slice over y now gives us the volume underneath the surface.

1.1 Changing order of integration

Although the order of integration does not make a difference in the final value of the double integral, it can make a big difference in how easy it is to evaluate the integral analytically. For example

$$\int_0^1 \int_0^2 xe^{xy} \mathrm{d}x\mathrm{d}y$$

Technically, we can integrate with respect to x first holding y constant using integration by parts. However, it seems much easier to evaluate the integral with respect to y first instead.

$$\int_0^2 \int_0^1 xe^{xy} \mathrm{d}y\mathrm{d}x = \int_0^2 xe^{xy} \,]_0^1 \,\mathrm{d}x = \int_0^2 e^x \mathrm{d}x = e^2 - 1$$

Sometimes, the integral might be impossible to evaluate analytically one way. Take

$$\int_1^2 \int_1^2 \frac{1}{y} e^{\frac{x}{y}} \,\mathrm{d}y\mathrm{d}x$$

This integral is impossible to evaluate the way it is written. Calculating it the other way however,

$$\int_1^2 \int_1^2 \frac{1}{y} e^{\frac{x}{y}} \,\mathrm{d}x\mathrm{d}y = \int_1^2 \int_1^2 \frac{1}{y^2} (e^{\frac{2}{y}} - e^{\frac{1}{y}}) \,\mathrm{d}y = (e^{\frac{1}{y}} - \frac{1}{2} e^{\frac{2}{y}})]_1^2 = e^{\frac{1}{2}} - \frac{3}{2}e + \frac{1}{2}e^2$$

1.2 Integrating over more general domains

While integrating over rectangles is quite useful, we have a much greater variety of domains that we did in single variable calculus. All domains of integration in single variable calculus had to be intervals of the form [a,b], but in 2-dimensions, we have rectangles, circles, ellipses, and so on which give much greater variety of domains to integrate over. Figure 2 from Rogawski sums up how to deal with the situation fairly well.

If we make the rectangles very small so that they approximate the domain very well, we can see that the top value of the rectangle will be the top function $g(x)$ and the bottom value the rectangles will reach will be the bottom function $h(x)$. Therefore, the sum of the integrals of some function $f(x, y)$ over this domain becomes

$$\iint f(x, y)\mathrm{d}A = \int_a^b \int_{h(x)}^{g(x)} f(x, y)\mathrm{d}y\mathrm{d}x$$

Here, a and b are the smallest and largest values x can take respectively and still remain inside the domain. The function $g(x)$ and $h(x)$ take care of the values that y can take inside the domain. Notice that this integral eventually evaluates to a number, so swapping the integral in this case to somehow put $g(x)$ and $h(x)$ on the outside and integrate with respect to x first does not really make sense. Instead, we have to find the upper and lower bounds on x as a function of y and then use those as the bounds (more on this later). It would be easiest to see these integrals evaluated with an example:

$$\int_0^1 \int_0^x 1\mathrm{d}y\mathrm{d}x$$

This domain is described by the picture in Figure 3.

The double integral gives the area of the triangle, since it gives the volume under the surface of height 1, which gives the same numerical value as the area. Doing the integral, we get

$$\int_0^1 \int_0^x 1\mathrm{d}y\mathrm{d}x = \int_0^1 y]_0^x \mathrm{d}x = \int_0^1 x\mathrm{d}x = \frac{1}{2}$$

We know the triangle represents the bounds on the integral because y goes from 0 to x, which is the boundary $y = x$ and x goes from 0 to 1. If we wanted to switch the bounds, we need the bounds on x as a function of y first and then the overall bounds on y. Looking at Figure 3, the largest x can become is 1. However, the smallest x can become is y. The values y can take also range between 0 and 1. Thus, we have

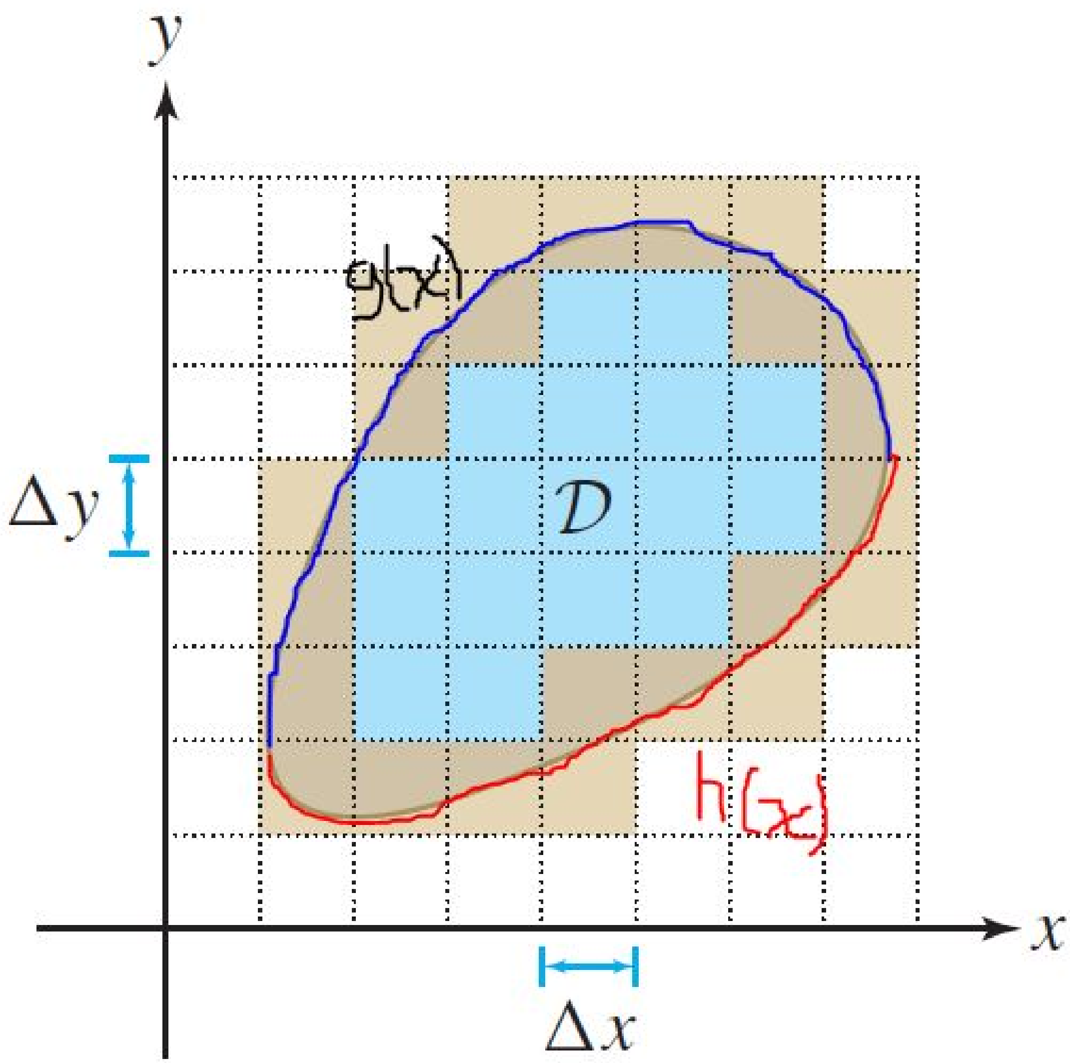

Figure 2: The graph of an arbitrary domain approximated using rectangles. (Graph adapted from Rogawski, 3rd edition, pg. 849)

$$\int_0^1 \int_0^x 1 \mathrm{d}y\mathrm{d}x = \int_0^1 \int_y^1 1 \mathrm{d}x\mathrm{d}y = \int_0^1 x]_y^1 \mathrm{d}y = \int_0^1 1 - y\mathrm{d}y = \frac{1}{2}$$

2 Triple Integrals

After learning double integrals, triple integrals largely use the same tools, but simply add an extra step of integration. If double integrals are defined over an area domain, triple integrals are defined over a volume domain which means it is not quite possible to visually see the value being calculated, but it is still useful to see the domain graphically. We can evaluated triple integrals over boxes and more general domains just as before:

$$\int_0^1 \int_0^1 \int_0^1 x^2 + y^2 + z^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \int_0^1 \frac{1}{3}x^3 + xy^2 + xz^2]_0^1 \mathrm{d}y\mathrm{d}z = \int_0^1 \int_0^1 \frac{1}{3} + y^2 + z^2 \mathrm{d}y\mathrm{d}z$$

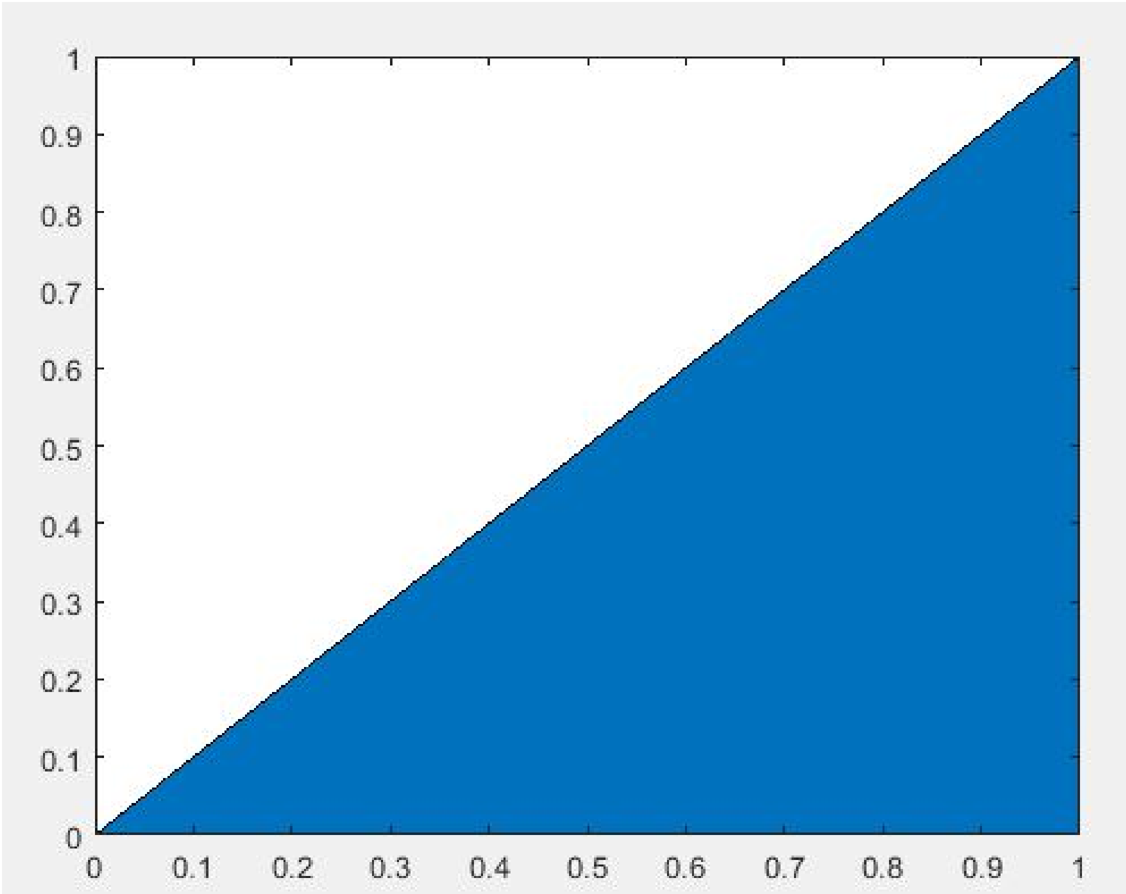

Figure 3: The domain of integration $\{(x, y) \in [0, 1] \times [0, 1] : y \leq x\}$. (Graph produced via Matlab)

At this point, we can solve the triple integral just as a double integral.

$$\int_0^1 \int_0^1 \frac{1}{3} + y^2 + z^2 \mathrm{d}y\mathrm{d}z = \int_0^1 \frac{2}{3} + z^2 \mathrm{d}z = 1$$

A sometimes useful trick in both the double and triple integral case is integrating over a box where the integrand is a product of single variable functions. In that case, we have

$$\int_e^f \int_c^d \int_a^b f(x)g(y)h(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \left(\int_a^b f(x)\mathrm{d}x\right) \int_e^f \int_c^d g(y)h(z)\mathrm{d}y\mathrm{d}z = \left(\int_a^b f(x)\mathrm{d}x\right)\left(\int_c^d g(y)\mathrm{d}y\right)\left(\int_e^f h(z)\mathrm{d}z\right)$$

This allows us to integrate a product of single variable functions as just a product of the single variable integrals. An example is shown below.

$$\int_1^2 \int_2^3 \int_0^1 xy^3z^2\mathrm{d}x\mathrm{d}y\mathrm{d}z = \left(\int_1^2 z^2\mathrm{d}z\right)\left(\int_2^3 y^3\mathrm{d}y\right)\left(\int_0^1 x\mathrm{d}x\right) = \left(\frac{7}{3}\right)\left(\frac{77}{4}\right)\left(\frac{1}{2}\right) = \frac{539}{8}$$

As an example of more complicated domains, consider the volume given in Figure 4.

If we want to calculate the integral

$$\iiint_W x\mathrm{d}V$$

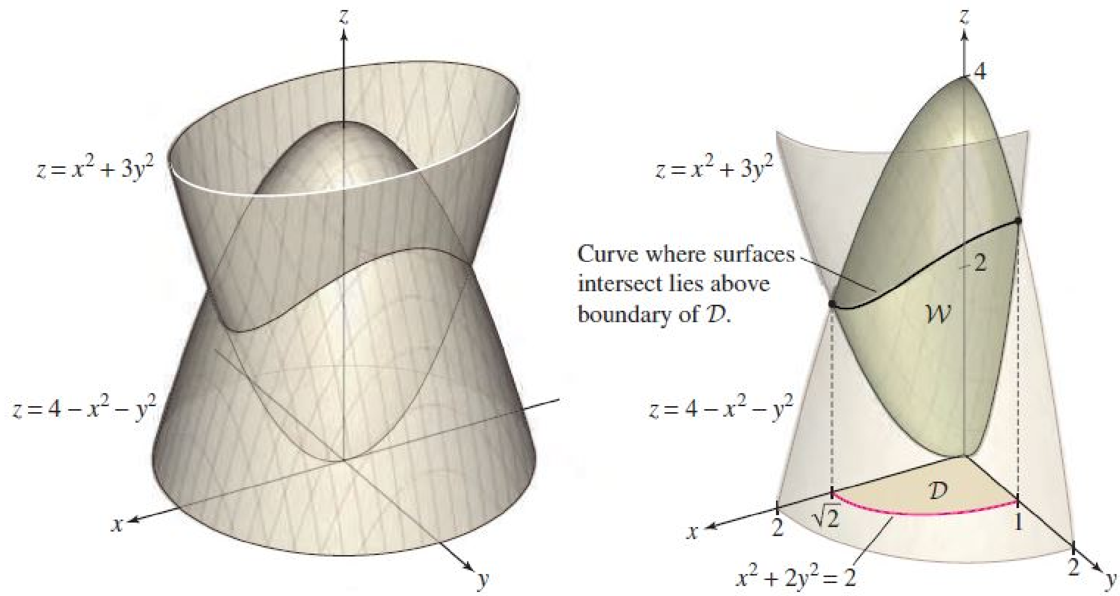

Figure 4: The domain of integration, W, taken from Rogawski, 3rd edition, page 865.The domain is the volume found in between the surfaces which is over the first quadrant in the x-y plane.

then choosing a good order of integration would do wonders for easiness of the calculation. Since the volume is bounded by the two surfaces which give the bounds on z as functions of x and y, it makes sense to make z the inner bounds.

$$\iiint_W x\mathrm{d}V = \iint_D \int_{x^2+3y^2}^{4-x^2-y^2} x\mathrm{d}z\mathrm{d}A$$

For the outer bounds, we need to see the largest values that x and y can take while still remaining inside the domain. This is given by the region where the two surfaces intersect, where

$$x^2 + 3y^2 = 4 - x^2 - y^2 \iff x^2 + 2y^2 = 2$$

This domain gives the ellipse shown in Figure 5.

The domain is fairly symmetric, so we can choose to either find the bounds for x in terms of y or vice versa. We choose x in terms of y because the bounds of y are simpler and the this will yield even polynomial orders of y after the x integration, which will be nice for cancelling out the square root. The bounds on x are then

$$0 \le x \le \sqrt{2 - 2y^2}$$

and the bounds on y are $[0, 1]$.

Thus, the integral is

$$\iiint_W x\mathrm{d}V = \int_0^1 \int_0^{\sqrt{2-2y^2}} \int_{x^2+3y^2}^{4-x^2-y^2} x\mathrm{d}z\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{\sqrt{2-2y^2}} x\left(4 - 4y^2 - 2x^2\right)\mathrm{d}x\mathrm{d}y$$

$$= \int_0^1 \int_0^{\sqrt{2-2y^2}} 4x\left(1-y^2\right)-2x^3\mathrm{d}x\mathrm{d}y = \int_0^1 2x^2\left(1-y^2\right)-\frac{1}{2}x^4\Big]_0^{\sqrt{2-2y^2}} \mathrm{d}y = \int_0^1 2\left(2-2y^2\right)\left(1-y^2\right)-\frac{1}{2}\left(2-2y^2\right)^2\mathrm{d}y$$

It is the not the point of this class to solve complicated single integrals, but in case you are interested, this simply involves quite a bit of expansion and polynomial integration.

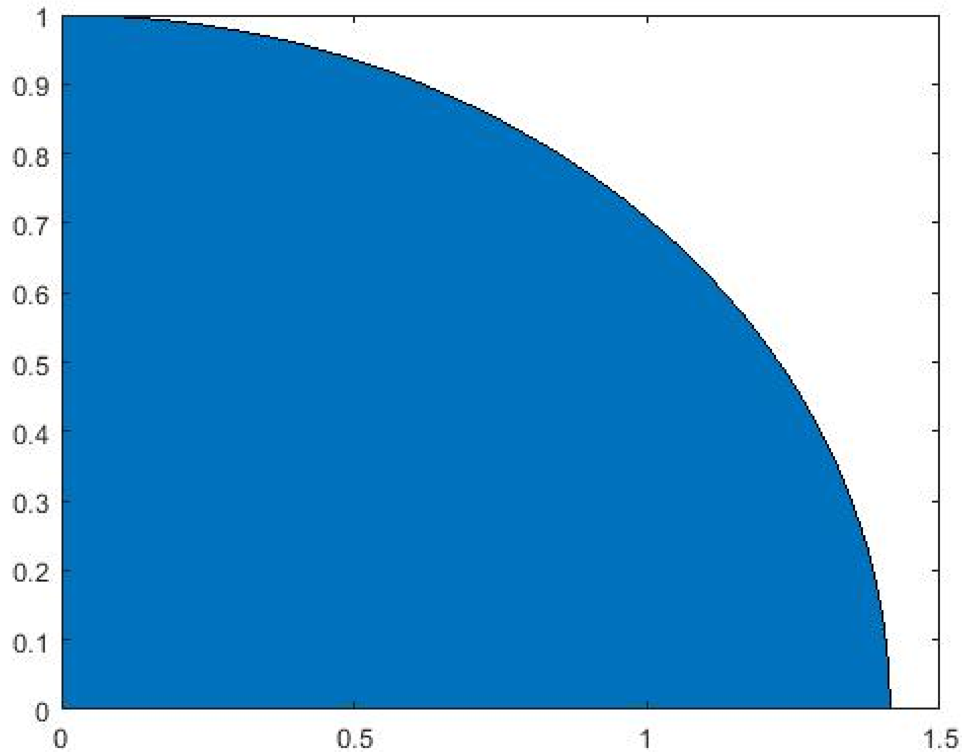

Figure 5: The outer domain, D, which describes the bounds on x and y. (Produced via Matlab)

$$\int_0^1 2\Big(2-2y^2\Big)\Big(1-y^2\Big)-\frac{1}{2}\Big(2-2y^2\Big)^2\mathrm{d}y=\int_0^1\Big(4-8y^2+4y^4\Big)-\Big(2-4y^2+2y^4\Big)\mathrm{d}y=\int_0^1 2-4y^2+2y^4\mathrm{d}y$$

$$=2y-\frac{4}{3}y^3+\frac{2}{5}y^5\Big]_0^1=2-\frac{4}{3}+\frac{2}{5}=\frac{2}{3}+\frac{2}{5}=\frac{16}{15}$$

We were fortunate enough to avoid trigonometric substitution in the last problem and as it happens, trigonometric substitution very rarely shows up in multiple integration due to our ability to make variable substitutions instead.

2.1 Change of Variables

The proof of change of variables is something done using linear algebra and so is left out of this class, but an example explains the concept fairly well.

Take the parallelogram displayed in Figure 6 as the domain. We would like to calculate

$$\iint_D 1\mathrm{d}A$$

Computing this integral with respect to either x or y would be messy as either the upper bound or the lower bound would change twice, leading to the sum of three double integrals, something we would all like to avoid doing. Consider the 32A way of calculating the area given by two parallelograms. It was calculated using the magnitude of the cross product of the two vectors that made up the sides, in this case,

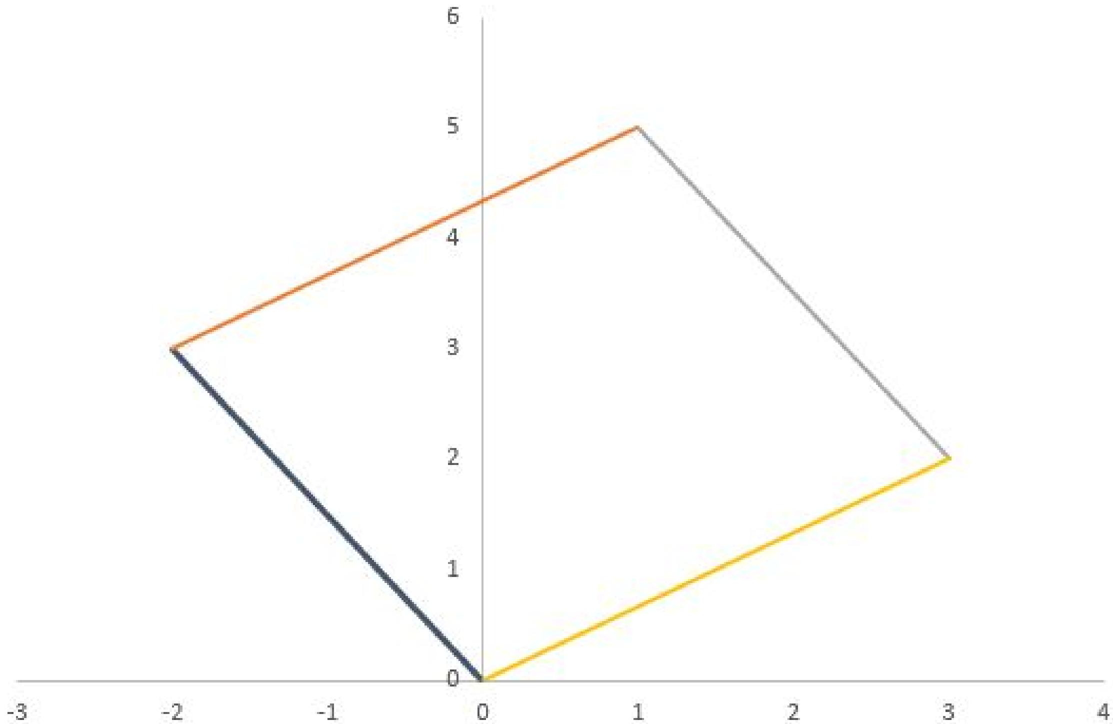

Figure 6: The inside of the parallelogram represents the domain, D, of integration. The area is not shaded to emphasize the boundary instead. (Produced via Excel)

$$A = \left| \det \left(\begin{bmatrix} -2 & 3 \\ 3 & 2 \end{bmatrix} \right) \right| = 13$$

Because the two vectors are $\vec{v_1} = \langle -2, 3 \rangle$ and $\vec{v_2} = \langle 3, 2 \rangle$.

Looking at it differently, let us define two new coordinates, u and v in terms of x and y given by the following formulas.

$$x = -2u + 3v$$

$$y = 3u + 2v$$

This can also be written in terms of the matrix above as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Then, $u = 0$, $v = 0$ gives the origin in (x, y); $(u, v) = (1, 0)$ gives $(x, y) = (-2, 3)$, $(u, v) = (0, 1)$ gives $(x, y) = (3, 2)$ and $(u, v) = (1, 1)$ gives $(x, y) = (1, 5)$, which are all the 4 vertices of the parallelogram defined above. Therefore, the domain of integration in terms of (u,v) is that in Figure 7, a square of sides 1.

So, the integral becomes simply the area of the square, 1, multiplied by the determinant which appropriately scales this value to get the value of the original integral. In general, a change of coordinates in 2-dimensions is done through expression x and y as functions of u and v and then finding the Jacobian matrix of the transformation,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g(u, v) \\ h(u, v) \end{bmatrix} \implies Jac = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

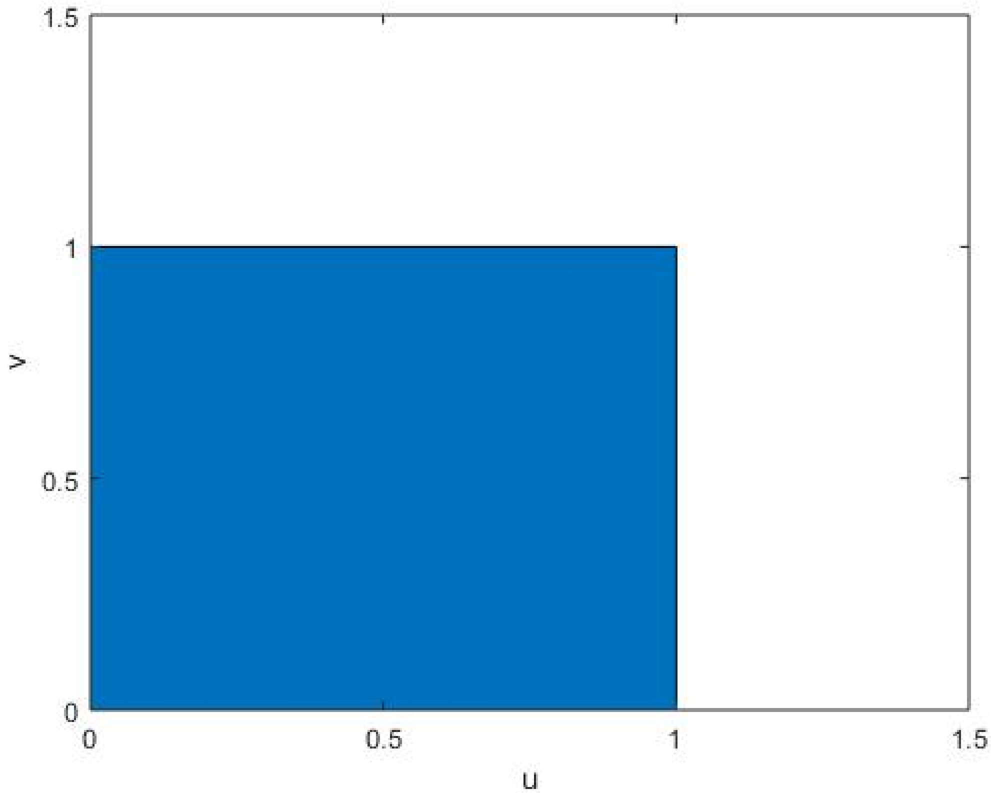

Figure 7: The transformation above defines the following square of sides 1 in (u,v) coordinates which is now the new domain, D'. (Produced via Matlab)

Then, the double integral $\iint_D f(x, y)\mathrm{d}x\mathrm{d}y$ can be found using

$$\iint_D f(x, y)\mathrm{d}x\mathrm{d}y = \iint_{D'} f(x(u, v), y(u, v))\,|Jac(u, v)|\mathrm{d}u\mathrm{d}v$$

where the $|Jac(u, v)|$ is the (absolute value) of the determinant of the Jacobian. This idea can be generalized to 3-dimensions using the 3-dimensional Jacobian instead.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} g(u, v, w) \\ h(u, v, w) \\ k(u, v, w) \end{bmatrix} \implies Jac = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

Then, the integral becomes

$$\iiint_D f(x, y, z)\mathrm{d}V(x, y, z) = \iiint_{D'} f(x(u, v, w), y(u, v, w), z(u, v, w)) * |Jac(u, v, w)|\,\mathrm{d}V(u, v, w)$$

It is again easiest to see this with an example, such as the area of an ellipse. Let us take a general ellipse given by the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

If we take the entire ellipse as the domain, we have to solve the following integral to get the area.

$$\iint_D 1\mathrm{d}A$$

Instead of solving this by expressing x in terms of y or vice versa, let us instead make the substitution $u = \frac{x}{a}$ and $v = \frac{y}{a}$. This turns the (u,v) domain into a circle of radius 1. In terms of x and y, $x = au$ and $y = bv$. This transformation is shown in Figure 8.

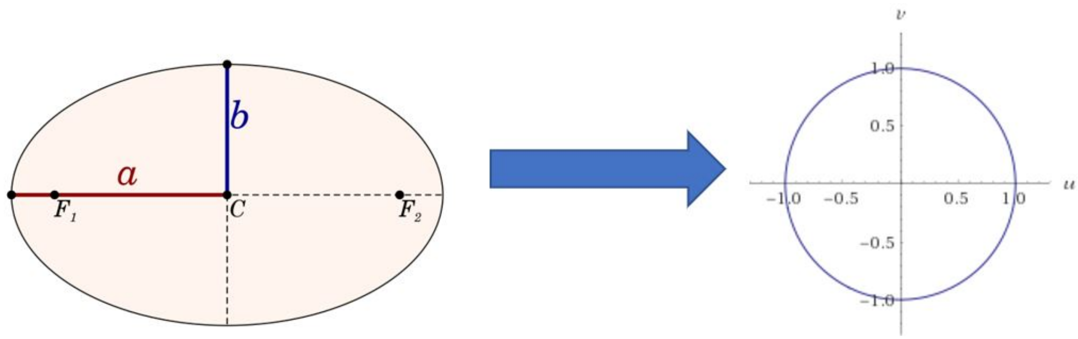

Figure 8: The defined transformation from an ellipse in (x,y) coordinates to the disk in (u,v) coordinates.

The determinant of the Jacobian is

$$\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

Then, the integral becomes

$$\iint_D 1 \mathrm{d}A(x,y) = \iint_{D'} ab \mathrm{d}A(u,v) = ab \iint_{D'} \mathrm{d}A(u,v) = \pi ab$$

Similarly, try showing that the volume of an ellipsoid which is surrounded by the surface $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ is πabc. It is very hard to know what kind of substitution will make your life easier, but there are a few common ones that are frequently useful. The one we just performed was a scaling of the axes. Another one, in 2-dimensions, is a change to polar coordinates, given by the transformation $x = r\cos(\theta)$ and $y = r\sin(\theta)$. This transformation has the Jacobian matrix determinant

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r$$

In 3-dimensions, we can define this same polar coordinate transformation while keeping the z-coordinate and thus using cylindrical coordinates; $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $z = z$.

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Finally, another transformation that is common is using spherical coordinates, $x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, and $z = \rho\sin(\phi)$. The physical interpretation of the symbols ρ, θ, and ϕ in this case can be seen in Figure 9.

The Jacobian in this case is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \cos(\theta)\sin(\phi) & -\rho\sin(\theta)\sin(\phi) & \rho\cos(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{vmatrix} = -\rho^2\sin(\phi)$$

In this case, the evaluation of the determinant is negative. However, this is simply a choice of the ordering of the coordinates since swapping two rows in a determinant flips the sign. Thus, we will take the absolute value of the determinant and get $|Jac| = \rho^2\sin(\phi)$.

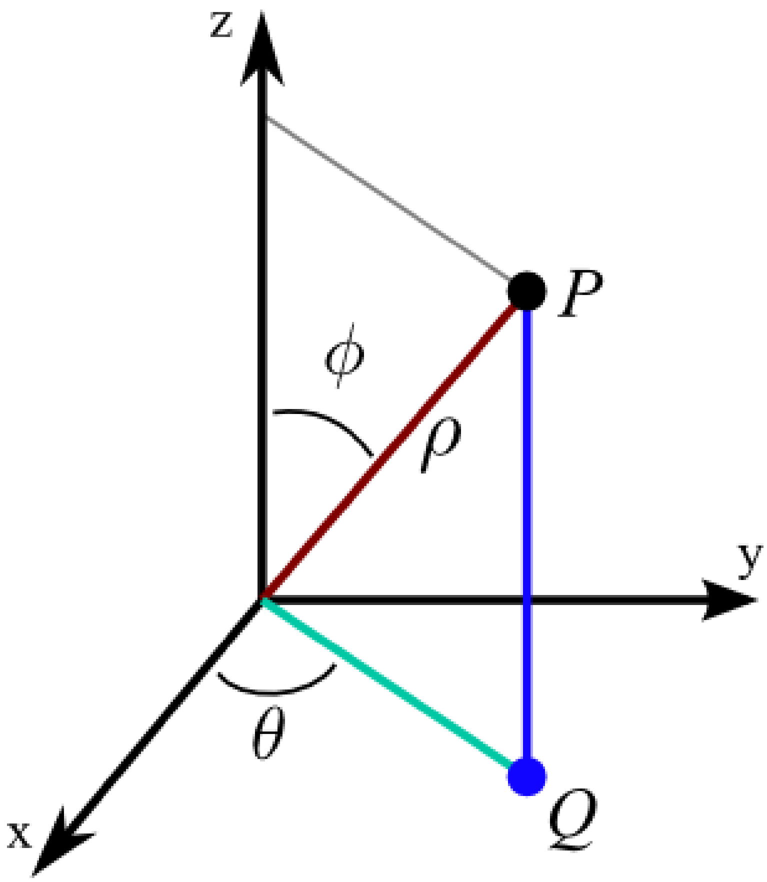

Figure 9: The ϕ angle indicates deviance from the vertical axis, the θ angle indicates rotation in the x-y plane as usual and ρ indicates distance from the origin.

As an example of spherical coordinates, consider the function $f(x, y, z) = (x^2 + y^2 + z^2)^{=frac32}$ with the domain $x^2 + y^2 + z^2 \leq 4$ and $z \leq \sqrt{x^2 + y^2}$. Then, the integral

$$\iiint_D f(x, y, z)dV$$

can be evaluated as

$$\iiint_{D'} \rho^{-\frac{3}{2}} * \rho^2 \sin(\phi)dV(\rho, \theta, \phi) = \iiint_{D'} \sqrt{\rho}\sin(\phi)dV(\rho, \theta, \phi)$$

We have to find the bounds in terms of ρ, θ, and ϕ. Since the original domain is a cone, ρ bounds simply go from 0 to 4 since there is no other upper and lower bound. Full rotation in the x-y plane is allowed, so θ bounds are 0 to 2π. Finally, for the ϕ bounds, we have that ϕ is bounded below by 0 and bounded above by the value of ϕ when $z = \sqrt{x^2 + y^2}$. Since $z = \rho\cos(\phi)$ and $\sqrt{x^2 + y^2} = \rho\sin(\phi)$, we end up with the equation

$$\tan(\phi) = 1$$

which can be solved to get $\phi = \frac{\pi}{4}$ as an upper bound.

Thus, the integral becomes

$$\iiint_{D'} \sqrt{\rho}\sin(\phi)dV(\rho, \theta, \phi) = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^4 \sqrt{\rho}\sin(\phi)d\rho d\phi d\theta = 2\pi\int_0^{\frac{\pi}{4}} \sin(\phi)d\phi \int_0^4 \sqrt{\rho}\sin(\phi)d\rho = 2\pi * \frac{2-\sqrt{2}}{2} * \frac{16}{3}$$

$$= \frac{16\pi * (2-\sqrt{2})}{3}$$

3 Line Integrals

Line integrals are a very special type of integral unlike the other ones we have seen before. Figure 10 presents the essential idea of how we view an integral.

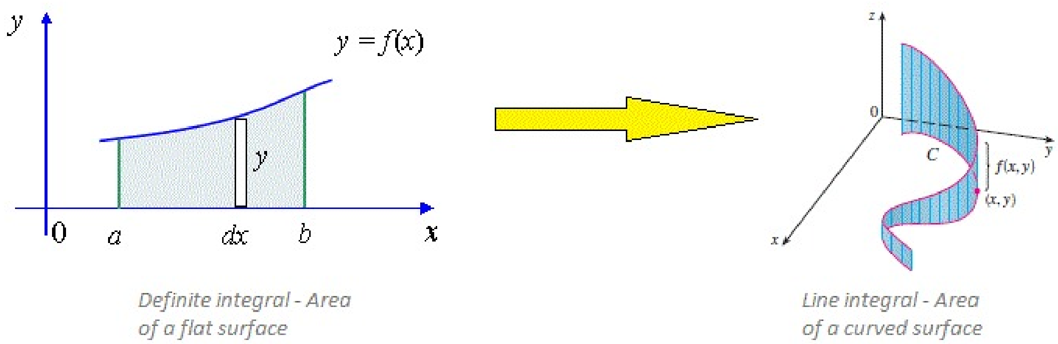

Figure 10: The physical intuition of an integral.

The integral finds the area, A, between an arbitrary function f(x) and the x axis between the x values of a and b. We sum up over dx, the differential element that describes how to change from a to b. In the line integral case, we instead have a surface $z = f(x, y)$ defined over a curve in the two dimensional case. The curve is in the x-y plane and the surface is above the x-y plane and denotes a height for every point along the curve. To find the area of the curtain that now drops from the surface to the curve, we sum up the areas of the rectangles, which have individual areas

$$\Delta A = f(x, y)\Delta s$$

where s describes the length of the rectangle, which is the arc-length of the segment of the curve and f(x,y) is the value of the surface which is also the height of the curve. Then, to find the area, we would need take the limit as all the Δs become small, which is the same as computing the integral over the curve, c.

$$\int_c f(x, y)\mathrm{d}s$$

This area is defined independent of a parametrization given to the curve since it is simply an intrinsic property of the curve and the surface over it. However, this integral is in practice computing by expressing both x and y in terms of a parameter and thus computing the integral as such. Let the parametrization be $\vec{r}(t) = \langle x(t), y(t)\rangle$ and the beginning and ending points of the integral be the points on the curve given by $\vec{r}(t_0)$ and $\vec{r}(t_1)$ respectively. Then,

$$\int_c f(x, y)\mathrm{d}s = \int_{t_0}^{t_1} f(x(t), y(t))\frac{ds}{dt}\mathrm{d}t$$

Since s described the arc length of the curve,

$$\frac{ds}{dt} = ||\vec{c}(t)|| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\int_c f(x, y)\mathrm{d}s = \int_{t_0}^{t_1} f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,\mathrm{d}t$$

Again, integrating the function $f(x, y) = 1$ will give you the length of the curve. As an example, let us integrate the function $f(x, y) = x$ along the path defined by the parabola given by $y = x^2$ from the point

$\langle 0,0\rangle$ to the point $\langle 1,1\rangle$. A parametrization of this curve can simply be given by

$$\langle x(t), y(t)\rangle = \langle t, t^2\rangle$$

Then,

$$\frac{ds}{dt} = \sqrt{1+4t^2}$$

The function in terms of the parameter t is

$$f(x(t), y(t)) = t$$

Thus, we have

$$\int_c f(x,y)\mathrm{d}s = \int_0^1 t\sqrt{1+4t^2}\mathrm{d}t = \frac{1}{8}\int_1^5 \sqrt{u}\mathrm{d}u = \frac{1}{8}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^5 = \frac{5^{\frac{3}{2}}-1}{12}$$

This idea extends nicely to three dimensions, which is beyond our ability to visualize, but still can be computed.

$$\int_c f(x,y,z)\mathrm{d}s = \int_{t_0}^{t_1} f(x(t),y(t),z(t))\frac{ds}{dt}\mathrm{d}t = \int_{t_0}^{t_1} f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\mathrm{d}t$$

As an example, consider the function $f(x,y,z) = x^2z$ over the path $\vec{r}(t) = \langle e^t, \sqrt{2}t, e^{-t}\rangle$ from $t=0$ to $t=1$. Then, we have

$$\int_c f(x,y,z)\mathrm{d}s = \int_0^1 e^t\sqrt{e^{2t}+2+e^{-2t}}\mathrm{d}t = \int_0^1 e^t\left(e^t+e^{-t}\right)\mathrm{d}t = \int_0^1 e^{2t}+1\mathrm{d}t = \frac{1}{2}e^{2t}+t\Big]_0^1 = \frac{e^2+1}{2}$$

The $\frac{ds}{dt}$ term in the line integral makes most integrals of this form impossible to compute analytically, but there is a special class of integrands that do not have this problem that are common; vector line integrals.

3.1 Vector Line Integrals

Before discussing a vector line integral itself, we first need to have a concept of a vector field. A vector field is a function that takes possibly multiple inputs and returns outputs as a vector. An example is $\vec{F}(x,y) = \langle x^2+y^2, xy\rangle$. In general, it is of the form $\vec{F}(x,y) = \langle f(x,y), g(x,y)\rangle$, or in three dimensions, $\vec{F}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z)\rangle$.

In two dimensions, this is equivalent to taking every point inside the domain inside the x-y plane and attaching a 2-dimensional vector to it. An example vector field, $\vec{F}(x,y) = \langle x,y\rangle$ is shown in Figure 11.

Another common example is the vector field given by $\vec{F}(x,y) = \langle y, -x\rangle$, which is Figure 12.

Finally, a more complex vector field, $\vec{F}(x,y) = \langle y, \sin(x)\rangle$, is shown in Figure 13. Notice how as x approaches 0, the arrows become horizontal as $\sin(0) = 0$ and as y approaches 0, the arrows becomes vertical.The arrows also flip from left to right moving right on the x=axis just as $\sin(x)$ flips from negative to positive.

Finally, a three-dimensional example is rarely drawn, but it is instructive to see at least one, shown in Figure 14 of $\vec{F}(x,y,z) = \langle x,y,z\rangle$.

A vector line integral is that which calculates

$$\int_c \left(\vec{F}\cdot\hat{T}\right)\mathrm{d}s$$

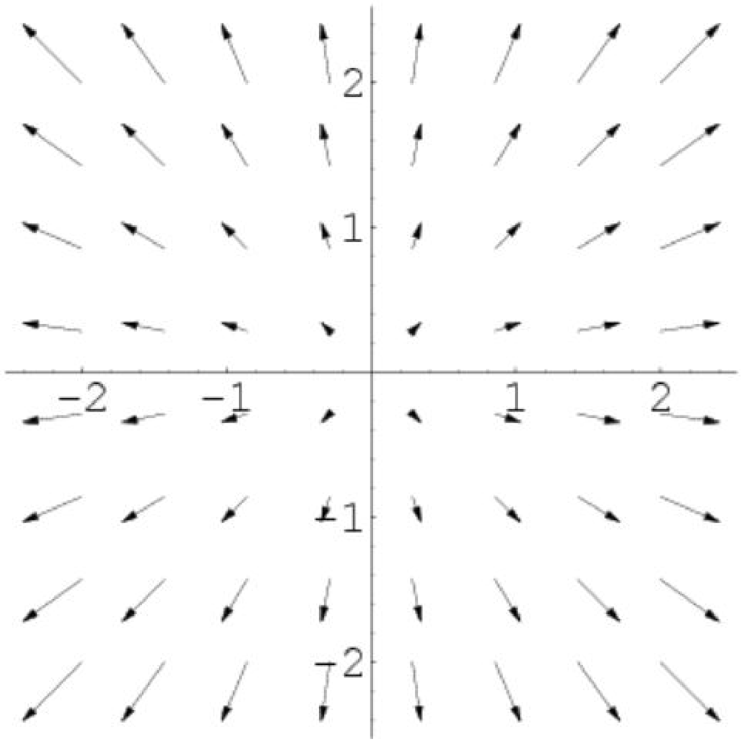

Figure 11: The vector field $\vec{F}(x,y) = \langle x, y\rangle$ is displayed. The length of the vectors correspond to the magnitudes.

where $\hat{T}$ is the unit tangent vector of the curve. This expression is still intrinsic to the curve and the vector field and not to the choice of parametrization, However, when we parametrize, we find the computation of this curve much simpler.

$$\int_c \left(\vec{F}\cdot\hat{T}\right)\mathrm{d}s = \int_{t_0}^{t_1}\left(\vec{F}\cdot\hat{T}\right)\frac{ds}{dt}\mathrm{d}t = \int_{t_0}^{t_1}\left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)\mathrm{d}t$$

This expression gets rid of the inherent square root that usually appears in the line integral. To denote that this expression is independent of the parametrization chosen, it is also written as

$$\int_c \vec{F}\cdot\mathrm{d}\vec{r}$$

As an example, let $\vec{F} = \langle xy, 2, z^3\rangle$ defined on the helix given by $\vec{r}(t) = \langle\cos(t), \sin(t), t\rangle$ from $t = 0$ to $t = \pi$. Then,

$$\vec{F}(\vec{r}(t)) = \langle\sin(t)\cos(t), 2, t^3\rangle$$

$$\frac{d\vec{r}}{dt} = \langle-\sin(t), \cos(t), 1\rangle$$

$$\int_c \vec{F}\cdot\mathrm{d}\vec{r} = \int_0^{\pi}\langle\sin(t)\cos(t), 2, t^3\rangle\cdot\langle-\sin(t), \cos(t), 1\rangle = \int_0^{\pi} -\sin^2(t)cos(t)+2\cos(t)+t^3\mathrm{d}t = -\frac{\sin^3(t)}{3}+2\sin(t)+\frac{t^4}{4}\bigg]_0^{\pi}$$

$$= \frac{\pi^4}{4}$$

As another example, consider the vector field $\vec{F} = \langle x, y\rangle$ integrated over the unit circle, or the path $\vec{r}(t) = \langle\cos(t), \sin(t)\rangle$ from $t = 0$ to $t = 2\pi$. We have

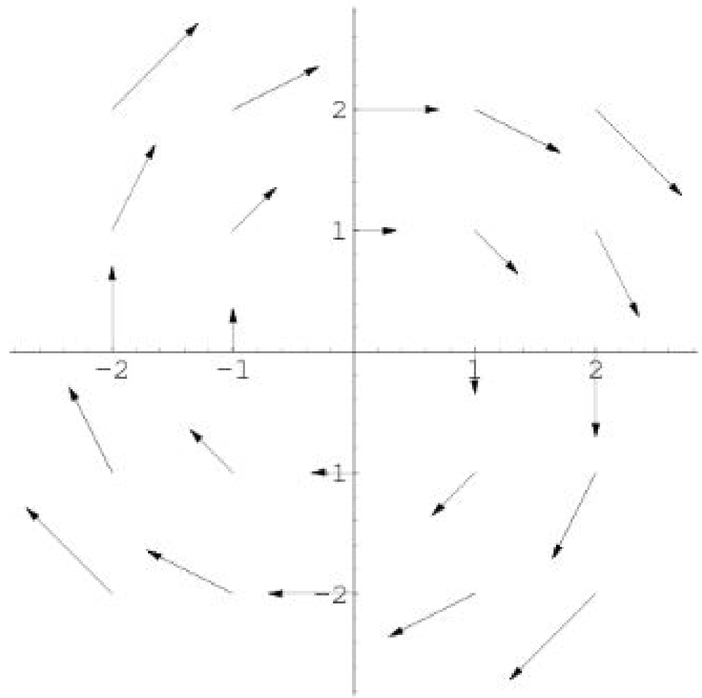

Figure 12: The vector field $\vec{F}(x, y) = \langle y, -x\rangle$ is displayed.

$$\vec{F}(\vec{r}(t)) = \langle \cos(t), \sin(t)\rangle$$
$$\frac{d\vec{r}}{dt} = \langle -\sin(t), \cos(t)\rangle$$
$$\int_c \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^{2\pi} \langle \cos(t), \sin(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle \mathrm{d}t = \int_0^{2\pi} 0 dt = 0$$

This answer somewhat makes sense because we have constructed a path that starts at a point, $\langle 1, 0\rangle$ and ends at the same point. However, this is actually not always the case, as we can see integrating the vector field $\vec{F} = \langle y, -x\rangle$. We instead get

$$\vec{F}(\vec{r}(t)) = \langle \sin(t), -\cos(t)\rangle$$
$$\int_0^{2\pi} \langle \sin(t), -\cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle \mathrm{d}t = \int_0^{2\pi} -1\mathrm{d}t = -2\pi$$

Therefore, the vector field determines whether line integrals around closed paths are 0. To denote a closed path, the following notation is common.

$$\oint_c \vec{F}\cdot \mathrm{d}\vec{r}$$

3.2 Conservative Vector Fields

When a vector field integrates such that its line integral is 0 for all closed loops, we call it a conservative vector field. This happens when the vector field is the gradient of some function f(x,y,z) because, from 32A,

$$\mathrm{d}f(\vec{r}(t)) = \vec{\nabla} f \cdot \frac{d\vec{r}}{dt}\mathrm{d}t$$

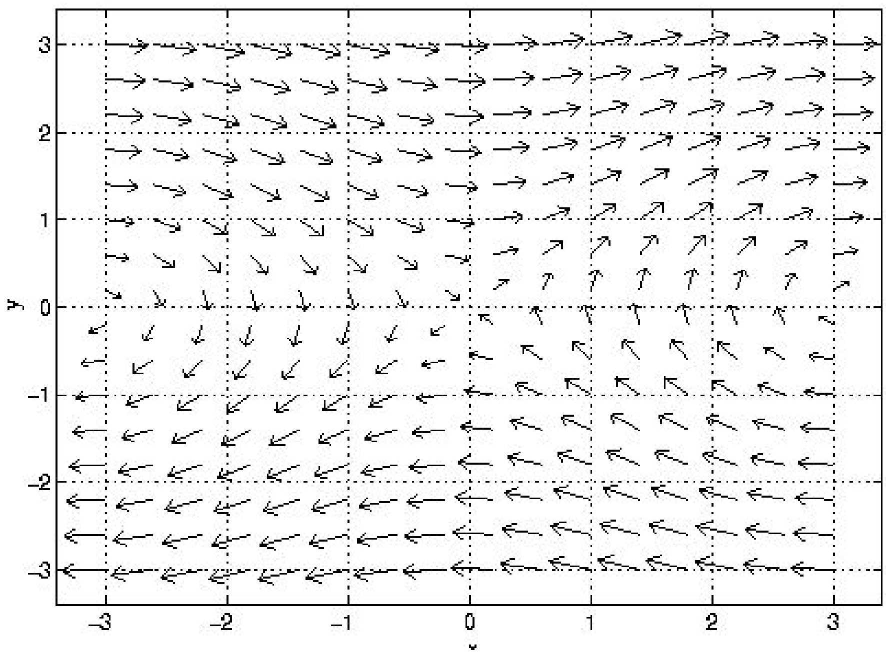

Figure 13: The vector field $\vec{F}(x,y) = \langle -y, x\rangle$ is displayed.

$$\oint_c \vec{F}\cdot \mathrm{d}\vec{r} = \int_{t_0}^{t_1} \vec{\nabla} f \cdot \frac{d\vec{r}}{dt}\mathrm{d}t = \int_{t_0}^{t_1} \mathrm{d}f(\vec{r}(t)) = f(\vec{r}(t))\Big]_{t_0}^{t_1} = 0$$

The last expression is 0 because the closed line integral implies that $\vec{r}(t_0) = \vec{r}(t_1)$.

A useful way of testing whether a vector field is the gradient of some function is checking the curl of the function, which checks all mixed derivatives. From 32A, we have

$$\frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}\Big) = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big)$$

so, for 2-dimensions, if $\vec{F} = \langle F_1(x,y), F_2(x,y)\rangle$ we must check

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial y}{\partial \theta} \\ F_1 & F_2 \end{vmatrix} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

For 3-dimensions, $\vec{F} = \langle F_1, F_2, F_3\rangle$ we must check

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\hat{i} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\hat{j} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\hat{i} = \langle 0,0,0\rangle$$

If this is the case, then we can find the function f itself and not bother with calculating the line integral directly. For example, let us calculate the line integral of $\vec{F} = \langle y, x, z^3\rangle$ over the path $\vec{r}(t) = \langle t, t^2, t^3\rangle$ from $t = 0$ to $t = 1$. In checking whether the field is conservative, we obtain

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & z^3 \end{vmatrix} = \big(0-0\big)\hat{i} + \big(0-0\big)\hat{j} + \big(1-1\big)\hat{i} = \langle 0,0,0\rangle$$

To find the field itself, let

$$\Big\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\Big\rangle = \langle y, x, z^3\rangle$$

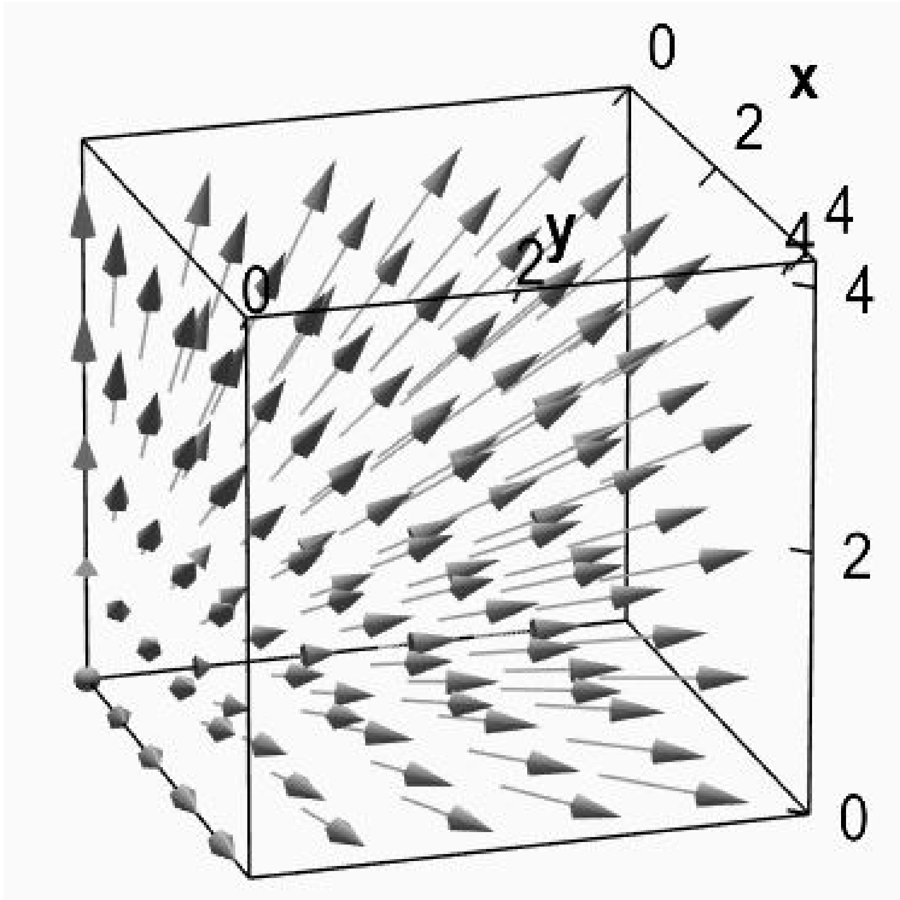

Figure 14: The vector field $\vec{F}(x, y, z) = \langle x, y, z \rangle$ is displayed.

$$\frac{\partial f}{\partial x} = y$$

Integrating with respect to x, we get

$$f(x, y, z) = xy + g(y, z)$$

Since differentiating with respect to x sends all functions of only y and z to 0, we obtain a constant of integration that is a function of y and z. We can use the other components of the vector field to find what this function is.

$$x + \frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} = x$$
$$\frac{\partial g}{\partial y} = 0$$
$$g(y, z) = h(z)$$
$$f(x, y, z) = xy + h(z)$$

Finally, we set the z derivative equal to the last component, z^3 to obtain what the expression for f is.

$$\frac{dh}{dz} = \frac{\partial f}{\partial z} = z^3$$
$$h(z) = \frac{z^4}{4} + C$$

$$f(x, y, z) = xy + \frac{z^4}{4} + C$$

The final C is truly a constant independent of x, y, and z. It will cancel out in the difference when doing the integral. Thus, for any path starting at $\langle 0, 0, 0\rangle$ and ending at $\langle 1, 1, 1\rangle$, the line integral for this vector field is

$$\int_c \vec{F} \cdot \mathrm{d}\vec{r} = f(1, 1, 1) - f(0, 0, 0) = \left(1 \cdot 1 + \frac{1^4}{4}\right) - \left(0 \cdot 0 + \frac{0^4}{4}\right) = \frac{5}{4}$$

4 Surface Integrals

We can also think about what it means to define an integral over a surface just as we did for a line. We do not have a visual representation in this case because surfaces are already 3-dimensional objects, but we can imagine a function that gives a value for every (x,y,z) coordinate on a surface. Then, we would somehow want to compute

$$\sum f(x, y, z)\Delta S$$

for all small surface area elements we divide the surface into and take the limit as ΔS goes to 0 to get

$$\iint_S f(x, y, z)\mathrm{d}S$$

The ds in the line integral case denoted the differential arc length element and so dS in this case must represent the differential surface area. From the example in Figure 6, we remember that the area of a parallelogram is given by the magnitude of the cross product of the vectors that make up the sides. Since the surface area can be approximated as parallelograms that become increasingly accurate as the surface area is divided up more and more, we can find dS as local cross product of the tangent vectors.

Suppose $\vec{r}(t, s) = \langle x(t, s), y(t, s), z(t, s)\rangle$ describes the surface. An example of this would be how $\vec{r}(x, y) = \langle x, y, x^2 + y^2\rangle$ describes the paraboloid $z = x^2 + y^2$. Then, the surface area element is given by the magnitude of the cross product of the tangent vectors multiplied by area element in terms of (t,s) coordinates, or

$$\mathrm{d}S = \left|\left|\frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial s}\right|\right| \mathrm{d}A(t, s)$$

$$\iint_S f(x, y, z)\mathrm{d}S = \iint f(x(t, s), y(t, s), z(t, s)) \left|\left|\frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial s}\right|\right| \mathrm{d}A(t, s)$$

As an example, let $f(x, y, z) = x + y + z$. Let us integrate this function over the surface of the cone $z^2 = x^2 + y^2$ from z = 0 to z = 4. The natural way to express the cone is in cylindrical coordinates, since we already have the equation $z = r$. Therefore, it seems natural to use $x = r\cos(\theta)$ and $y = r\sin(\theta)$ as our two parameters. The bounds on theta are $[0, 2\pi]$ in that case and for r, we have $0 \leq r = z \leq 4$, so the bounds on r are $[0, 4]$. Therefore, we have our parametrization $\vec{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), r\rangle$ and our r and θ bounds as specified above. Now we must calculate the cross product.

$$\frac{\partial \vec{r}}{\partial r} = \langle \cos(\theta), \sin(\theta), 1\rangle$$

$$\frac{\partial \vec{r}}{\partial \theta} = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$$

$$\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(\theta) & \sin(\theta) & 1 \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \langle -r\cos(\theta), -r\sin(\theta), r\rangle$$

$$\frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = \sqrt{2}r$$
$$f(x(r,\theta), y(r,\theta), z(r,\theta)) = \Big(r\cos(\theta)\Big) + \Big(r\sin(\theta)\Big) + (r) = r\Big(1+\sin(\theta)+\cos(\theta)\Big)$$
$$\iint_S f(x,y,z)\mathrm{d}S = \iint_D r\Big(1+\sin(\theta)+\cos(\theta)\Big)\cdot\sqrt{2}r dA(r,\theta) = \sqrt{2}\int_0^{2\pi}\int_0^4 r^2\Big(1+\sin(\theta)+\cos(\theta)\Big)\mathrm{d}r\mathrm{d}\theta$$
$$= \sqrt{2}\bigg(\int_0^4 r^2\mathrm{d}r\bigg)\bigg(\int_0^{2\pi}\Big(1+\sin(\theta)+\cos(\theta)\Big)\mathrm{d}\theta\bigg) = \sqrt{2}\cdot\frac{64}{3}\cdot 2\pi = \frac{128\pi\sqrt{2}}{3}$$

4.1 Vector Surface Integrals

Just as we previously defined a vector line integrals as taking the integrand as a vector field dotted with the unit tangent vector, we accomplish a similar task with a surface integral. It does not make sense to calculate the amount of a vector field is "parallel" to a surface, but it does make sense to compute how much of a vector field is normal to a surface, or $\vec{F}\cdot\hat{n}$. Then, a vector surface integral is

$$\iint_S \Big(\vec{F}\cdot\hat{n}\Big)\mathrm{d}S$$

Again, this quantity is independent of parametrization, but the parametrization greatly simplifies its calculation since $\vec{n} = \frac{\partial \vec{r}}{\partial t}\times\frac{\partial \vec{r}}{\partial s}$ is already perpendicular to the surface as it is found through the cross product of the tangent vectors. Then,

$$\iint_S \Big(\vec{F}\cdot\hat{n}\Big)\mathrm{d}S = \iint_S \Big(\vec{F}\cdot\hat{n}\Big)||\vec{n}||\mathrm{d}A(t,s) = \iint_S \vec{F}\cdot\vec{n}\mathrm{d}A(t,s)$$

As a shorthand, to declare that this value is independent of parametrization, it is often written as

$$\iint_S \vec{F}\cdot\mathrm{d}\vec{S}$$

A physical intuition that can be given to this type of integral is thinking of the vector field $\vec{F}$ as a field that describes the velocity of water at every point and S as a surface in the water. The value of $\iint_S \vec{F}\cdot\mathrm{d}\vec{S}$ describes how much water passes through the surface. As an example, let $\vec{F} = \langle y, x, z\rangle$ and let the plane $x+2y+z=1$ be the surface inside the first octant with an upward pointing normal. A parametrization that can be given to this surface is $\vec{r} = \langle x, y, 1-x-2y$. Then,

$$\frac{\partial \vec{r}}{\partial x} = \langle 1, 0, -1\rangle$$
$$\frac{\partial \vec{r}}{\partial y} = \langle 0, 1, -2\rangle$$
$$\frac{\partial \vec{r}}{\partial x}\times\frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{vmatrix} = \langle 1, 2, 1\rangle$$

This is the normal vector we would have gotten from just taking the coefficients in front of the variables as usual from a plane, but that shortcut only works if the plane equation is expressed in the form $ax+by+cz=1$.

$$\vec{F}(\vec{r}(x,y)) = \langle y, x, 1-x-2y\rangle$$
$$\iint_S \vec{F}\cdot\mathrm{d}\vec{S} = \iint_D \langle y, x, 1-x-2y\rangle\cdot\langle 1,2,1\rangle\mathrm{d}A(x,y) = \iint_D 1+x-y\mathrm{d}A(x,y)$$

The domain in this case is the area in the positive x-y plane where $0 \le 1-x-2y$ or where $y \le \frac{1-x}{2}$. Thus, the x bounds are $[0,1]$ and the y bounds are $[0, \frac{1-x}{2}]$.

$$\vec{F}(\vec{r}(x,y)) = \langle y, x, 1-x-2y\rangle$$
$$\iint_S \vec{F}\cdot\mathrm{d}\vec{S} = \int_0^1\int_0^{\frac{1-x}{2}}(1+x-y)\mathrm{d}y\mathrm{d}x = \int_0^1 \frac{1}{2}(1-x^2)-\frac{1}{4}(1-x)^2\mathrm{d}x = \frac{1}{4}\int_0^1(1+x)^2\mathrm{d}x = \frac{(1+x)^3}{3}\bigg]_0^1 = \frac{7}{3}$$

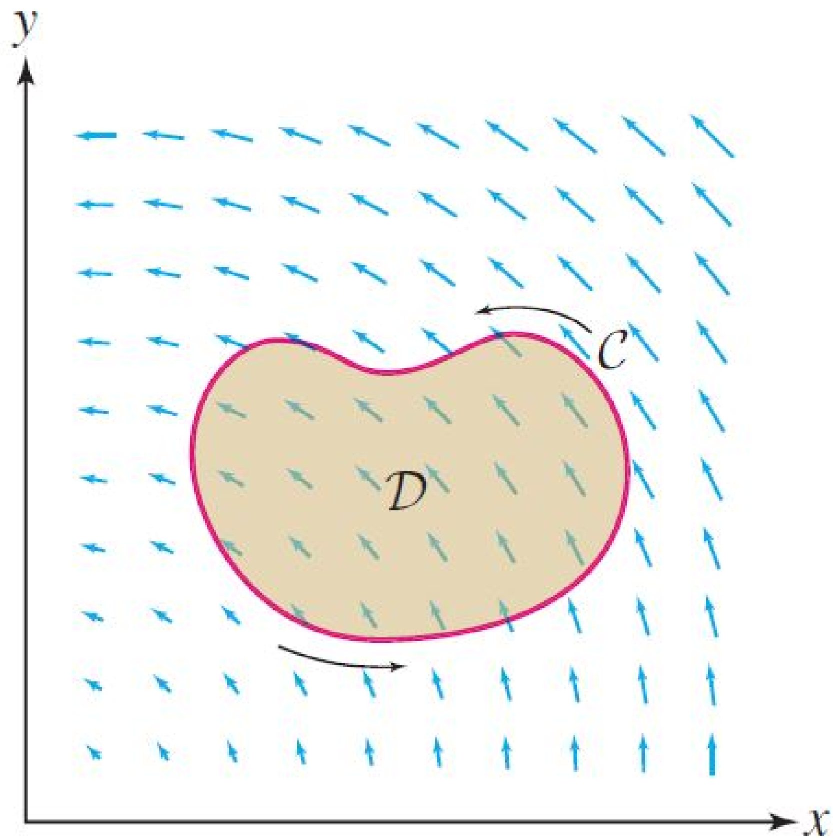

Figure 15: An area domain, D, is surrounded by a curve, c.

5 Fundamental Theorems of Multivariable Calculus

Similar to the fundamental theorem of calculus, we also have fundamental theorems of multivariable calculus. As a reminder, if $f(x) = F'(x)$, then we had

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

This gives us a way of turning a higher order (in this case 1st) integral into a lower (0th) order integral, which is just a difference. Green's theorem has the same idea in mind by turning a double integral over an area into a line integral around the boundary.

5.1 Green's Theorem

Green's theorem involves the situation present in Figure 15.

When computing the line integral of $\vec{F} = \langle F_1, F_2 \rangle$, $\oint_c \vec{F} \cdot \mathrm{d}\vec{r}$, we have the choice of doing it directly or instead evaluating it as a double integral through the formula

$$\oint_c \vec{F} \cdot \mathrm{d}\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A$$

which works for all simply connected (no holes in the domain or vector field) domains. As an example, consider the example we had before with the curve given by $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ from $t = 0$ to $t = 2\pi$. This closed loop surrounds the unit circle of area 1. We had two vector fields that we calculated this line integral for, $\vec{F_1} = \langle x, y \rangle$ and $\vec{F_2} = \langle y, -x \rangle$. We can see that

$$\oint_c \vec{F_1} \cdot \mathrm{d}\vec{r} = \iint_D \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \mathrm{d}A = 0$$

$$\oint_c \vec{F_2} \cdot \mathrm{d}\vec{r} = \iint_D \left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y} \right) \mathrm{d}A = \iint_D -2\mathrm{d}A = -2 \iint_D 2\mathrm{d}A = -2\pi$$

which is much easier to compute than the line integral case. One important thing to remember is that the direction that the curve is oriented is counterclockwise. Otherwise, the opposite direction must instead be taken by adding a minus in front of the line integral.

Another important thing to note is that the domain must be simply connected; neither the vector field nor the domain can have holes inside of it. As an example, take the vector field

$$\vec{F} = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

along the same unit circle path as before. We have

$$\frac{\partial F_2}{\partial x} - \frac{\partial x}{\partial F_1} = 0$$

In fact, this vector field is the gradient of the function $f(x, y) = \tan^{-1}(\frac{y}{x})$. However,

$$\vec{F}(\vec{r}(t)) = \langle -\sin(t), \cos(t) \rangle$$

$$\frac{r}{dt} = \langle -\sin(t), \cos(t) \rangle$$

$$\oint_c \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^{2\pi} \langle -\sin(t), \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \mathrm{d}t = \int_0^{2\pi} 1 dt = 2\pi$$

Therefore, Green's theorem fails in this case.

5.2 Stokes' Theorem

Stokes' theorem generalizes the ideas from Green's theorem to 3-dimensional line integrals. We cannot expect a 3-dimensional closed loop to surround an area inside the x-y plane anymore, so it instead relates the line integral to a surface integral of the curl instead.

$$\oint_{\partial S} \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S \left(\vec{\nabla} \times \vec{F} \right) \cdot \mathrm{d}\vec{S}$$

In Green's theorem, we were only able to pick one area that was enclosed by the closed loop; however, here, we can actually pick ANY surface which has the closed loop as its boundary. Figure 16 presents some usual cases.

For the first figure, we can easily apply Stokes' theorem as written. A few important points should be noted however. The direction of the surface integral can be oriented either way. However, the line integral will then be oriented in that direction. For example, with the surface on the left, if we want the normal vector to point outward, this signals that we should want the direction of the curve to be counterclockwise whereever the boundary touches the surface. Excusing a poor drawing, consider the same surface now in Figure 17.

Therefore, Stokes' theorem holds normally and we can write

$$\oint_{\partial S} \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S \left(\vec{\nabla} \times \vec{F} \right) \cdot \mathrm{d}\vec{S}$$

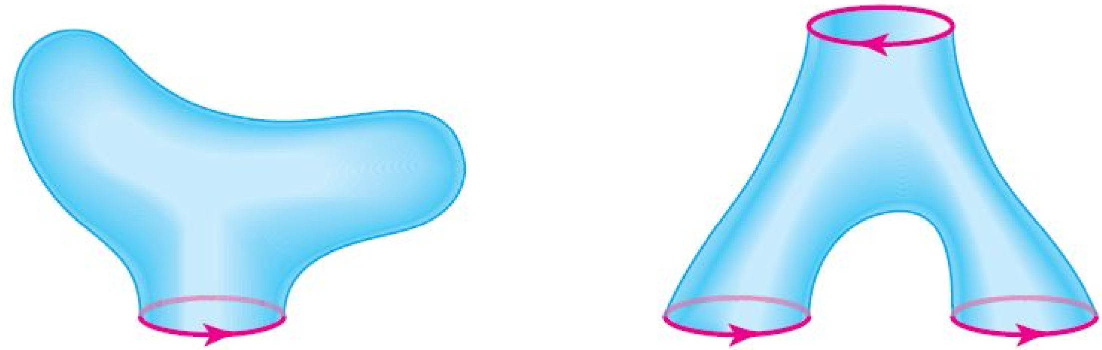

Figure 16: The first surface is bounded by only one curve, so we can directly apply Stokes' theorem. The second surface is instead bounded by three curves, so we must use a modified version of Stokes'.

If we instead wanted the normal vector to point inward and still keep the orientation of the line integral as it is, we would instead have

$$-\oint_{\partial S} \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S \left(\vec{\nabla} \times \vec{F}\right) \cdot \mathrm{d}\vec{S}$$

For the second surface, we use the same idea of orientation. Suppose we want the outward pointing surface integral again. Then, the drawing becomes that in Figure 18.

Thus, in this case, since all the curves point in the direction they should (even though it appears as if ∂S_3 is pointing in the "wrong" direction, we have

$$\iint_S \left(\vec{\nabla} \times \vec{F}\right) \cdot \mathrm{d}\vec{S} = \int_{\partial S_1} \vec{F} \cdot \mathrm{d}\vec{r} + \int_{\partial S_2} \vec{F} \cdot \mathrm{d}\vec{r} + \int_{\partial S_3} \vec{F} \cdot \mathrm{d}\vec{r}$$

As an example of Stokes' theorem, consider the portion of the plane $\frac{x}{2} + \frac{y}{3} + z = 1$ in the first octant. We would like to find the line integral around the boundary of this surface of the vector field $\vec{F} = \langle yz, 0, x\rangle$. The boundary is not very continous; computing the line integral directly would require doing 3 integrals. However, the surface integral is much more manageable. Recall that the normal vector for a surface integral of a plane of the form $ax + by + cz = 1$ is simply the vector $\langle a, b, c\rangle$. In our case, that is $\left\langle \frac{1}{2} \cdot \frac{1}{3}, 1 \right\rangle$. To use this normal vector, we must also use the parametrization $\vec{r}(x, y) = \langle x, y, \frac{1-ax-by}{c}\rangle$; in our case, $\vec{r}(x, y) = \langle x, y, 1 - \frac{x}{2} - \frac{y}{3}\rangle$. Then

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 0 & x \end{vmatrix} = \langle 0, y-1, -z\rangle = \left\langle 0, y-1, -1 + \frac{x}{2} + \frac{y}{3} \right\rangle$$

The bounds for the domain in terms of x and y are where $0 \le x$, $0 \le y$, and $\frac{x}{2} + \frac{y}{3} \le 1$. This gives the bounds for x as $[0, 2]$ and for y as $\left[0, 3 - \frac{3}{2}x\right]$.

$$\oint_{\partial S} \vec{F} \cdot \mathrm{d}\vec{r} = \iint_s \vec{F} \cdot \mathrm{d}\vec{S} = \int_0^2 \int_0^{3-\frac{3}{2}x} \left\langle 0, y-1, -1+\frac{x}{2}+\frac{y}{3} \right\rangle \cdot \left\langle \frac{1}{2}, \frac{1}{3}, 1 \right\rangle \mathrm{d}y\mathrm{d}x = \int_0^2 \int_0^{3-\frac{3}{2}x} \left(\frac{1}{2}x + \frac{2}{3}y - \frac{4}{3}\right) \mathrm{d}y\mathrm{d}x$$

$$= \int_0^2 \left(\frac{1}{2}x - \frac{4}{3}\right)\left(3 - \frac{3}{2}x\right) + \frac{1}{3}\left(3 - \frac{3}{2}x\right)^2 \mathrm{d}x = \int_0^2 \frac{1}{2}x - 1\mathrm{d}x = \frac{1}{4}x^2 - x\Big]_0^2 = -1$$

5.3 Divergence Theorem

Finally, we have come to the last fundamental theorem of multivariable calculus, Divergence Theorem (also known as Gauss's theorem). Divergence theorem uses a similar idea to Green's theorem and Stokes' theorem

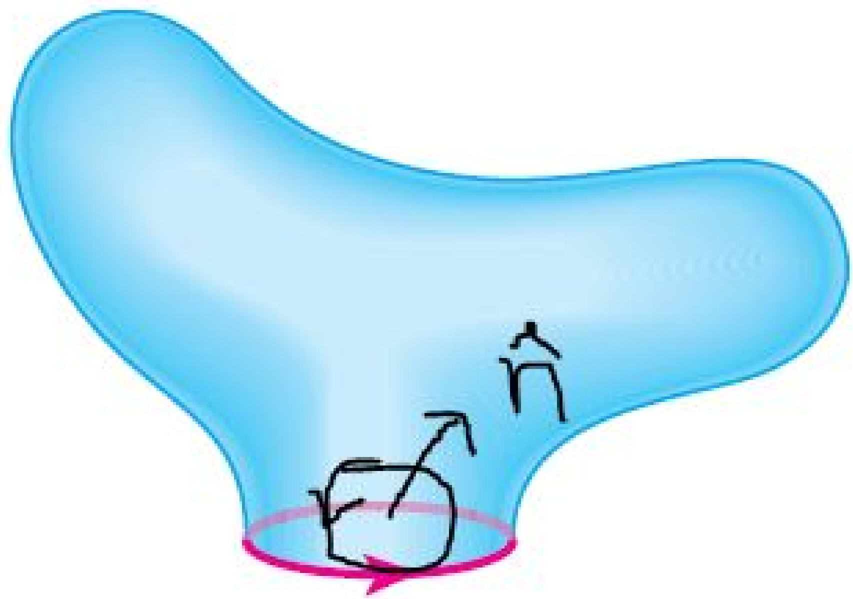

Figure 17: An outward pointing vector prompts a counterclockwise orientation for line integrals bounding the surface; the line integral here is oriented correctly because the arrows on the normal vector and curve point along the same direction where they touch.

by relating a surface integral to a volume integral. Before that, however, we first need the definition of divergence of a vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle$.

The divergence of a vector field, $\vec{\nabla} \cdot \vec{F}$, is defined as

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Then, by divergence theorem, we have, for a volume, W, and the surface that bounds it, ∂W,

$$\oiint_{\partial W} \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_W \left(\vec{\nabla} \cdot \vec{F}\right) \mathrm{d}V$$

There are surprisingly few nuances with this definition and the only thing to really keep track of is the fact that the surface must be oriented outwards the way this definition is written. Also, the domain must be simply connected and the vector field must be defined everywhere on the inside just like Green's theorem. Let us see an example where we compute it both ways.

Consider the vector field $\vec{F} = \langle x, y, z \rangle$ and the surface given by $x^2 + y^2 + z^2 = 1$ that encloses the unit ball centered around the origin. Let us compute the surface integral first. A parametrization of the surface is given from spherical coordinates,

$$\vec{r}(\theta, \phi) = \langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) \rangle$$

Then,

$$\frac{\partial \vec{r}}{\partial \theta} = \langle -\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0 \rangle$$

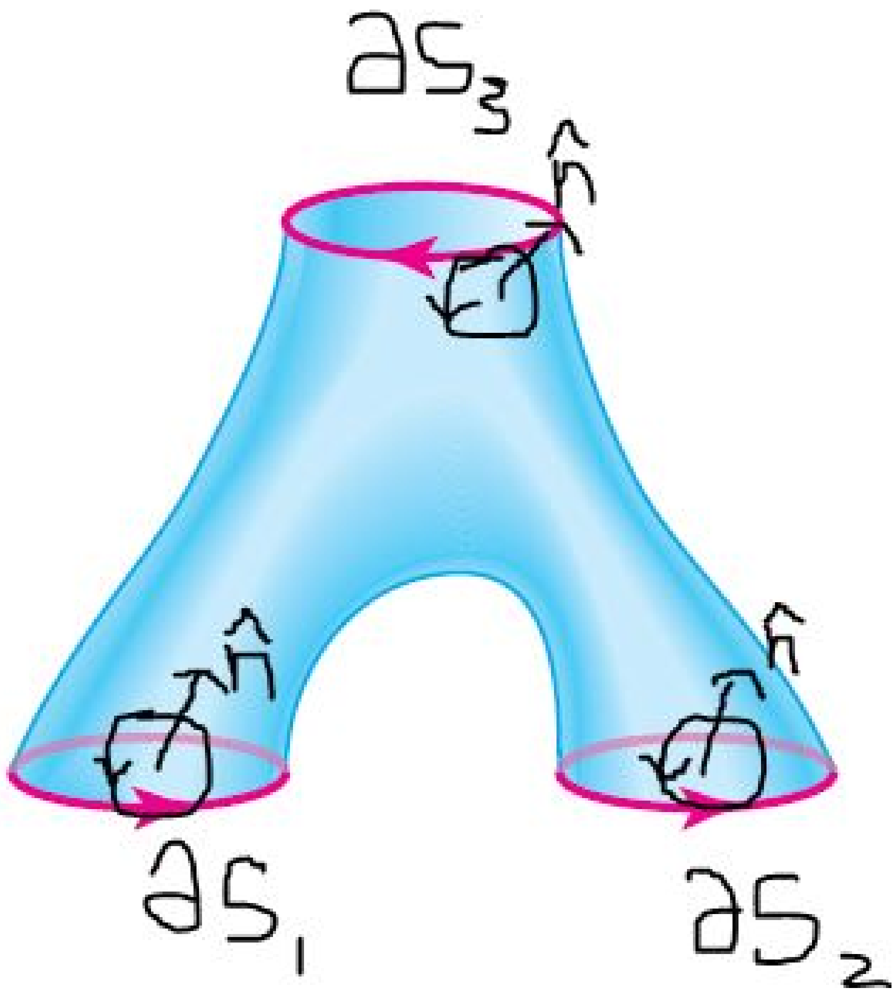

Figure 18: An outward pointing vector prompts a counterclockwise orientation for line integrals bounding the surface; all line integrals here are oriented correctly because the arrows on the normal vectors and curves point along the same direction where they touch.

$$\frac{\partial \vec{r}}{\partial \phi} = \langle \cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi) \rangle$$

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi) & 0 \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi) \end{vmatrix} = \langle -\sin^2(\phi)\cos(\theta), -\sin^2(\phi)\sin(\theta), -\sin(\phi)\cos(\phi) \rangle$$

It is hard to tell if this normal vector is pointing outward or not, so we can test it by checking a single point; $\langle \theta, \phi \rangle = \langle 0, \frac{\pi}{2} \rangle$.

$$\left\langle -\sin^2\left(\frac{\pi}{2}\right)\cos(0), -\sin^2\left(\frac{\pi}{2}\right)\sin(0), -\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) \right\rangle = \langle -1, 0, 0 \rangle$$

So the normal vector is pointing inward, which means we really want the negative of this vector, so

$$\vec{n} = \langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi) \rangle$$

Notice we did not try $\langle \theta, \phi \rangle = \langle 0, 0 \rangle$ instead as would be anyone's first try. It gives a normal vector of $\langle 0, 0, 0 \rangle$ which does not tell us anything about the direction,

$$\vec{F}(\vec{r}(\theta, \phi)) = \langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) \rangle$$

$$\oiint_{\partial W} \vec{F} \cdot \mathrm{d}\vec{S} = \int_0^{2\pi} \int_0^{\pi} \langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) \rangle \cdot \langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi) \rangle \mathrm{d}\phi \mathrm{d}\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \sin(\phi) \mathrm{d}\phi \mathrm{d}\theta = 2\pi \int_0^{\pi} \sin(\phi) \mathrm{d}\phi = 4\pi$$

Now that we have done this the hard way, let us see the easy way to do it.

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} = 3$$

$$\oiint_{\partial W} \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_W \left(\vec{\nabla} \cdot \vec{F}\right) \mathrm{d}V = \iiint_W 3 \mathrm{d}V = 3 \iiint_W \mathrm{d}V = 3 * \left(\frac{4}{3}\pi\right) = 4\pi$$

This is not to say that you can always get away with a volume integral being easier than the surface integral, but it usually is easier to just find the bounds on a volume than to go through finding a parametrization of a surface.