Math 33B Review Sheet

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Contents

1 First Order Differential Equations

Ordinary differential equation (ODE): a differential equation that has only one independent variable and all derivatives are with respect to this variable
Examples $y' = x^2$, $sin(xy') = y$, $(y')^2 = xy$

Partial differential equation (PDE): a differential equation which has derivatives with respect to more than one independent variable
Examples: $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ (wave equation), $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ (heat equation)

1.1 Separable

Appears as: $f(x)dx = g(y)dy$
Integrate both sides to obtain an implicit solution
$\int f(x)dx = \int g(y)dy + C$
$F(x) = G(y) + C$

Ex: $y\,dy = \sin(x)\,dx$

This is exactly in the form $f(x)dx = g(y)dy$ with $f(x) = \sin(x)$, $g(y) = y$
Therefore, we can integrate both sides to get the following. $\int y\,dy = \int sin(x)\,dx$
$\frac{y^2}{2} = -\cos(x) + C$
$y^2 = -2\cos(x) + C$
The 2 was multiplied to both sides, but since the C is an arbitrary constant, 2C is also arbitrary, so it is swallowed inside the constant.
$y = \pm\sqrt{-2\cos(x) + C}$
The actual sign of the solution will need to be determined using the initial condition; suppose the initial condition is (0,1).
$1 = \pm\sqrt{-2\cos(0) + C}$
$1 = \pm\sqrt{-2 + C}$
A square root cannot be negative, so the sign of the square root must be positive. Also, for the solution to work, the constant of integration must be 3. Thus, the solution is
$y = \sqrt{-2\cos(x) + 3}$

1.2 Linear Homogeneous

Appears as: $y' = a(x)y$
A general explicit solution may be found by separation

$\int \frac{dy}{y} = \int adx + c$
$ln|y| = \int adx + c$
$y = Ae^{\int adx}$

Ex: $y' = \frac{y}{x^2+1}$
Here, the a(x) mentioned before is not $x^2 + 1$, but rather $\frac{1}{x^2+1}$.
$\frac{dy}{y} = \frac{dx}{x^2+1}$
$\int \frac{dy}{y} = \int \frac{dx}{x^2+1}$
$ln|y| = \tan^{-1}(x) + C$
$y = e^{\tan^{-1}(x)+C} = Ce^{\tan^{-1}(x)}$
Here, the constant C has once again been relabeled such because it is arbitrary and so is e^C. Also not that in this case, although the constant C has been relabeled from e^C, it need not be positive. This is because the

equation originally solved was $\ln|y| = \tan^{-1}(x) + C$, so y is allowed to be negative if need be. The solution y may also be the zero function in fact, as this also solves the differential equation $y' = \frac{y}{x^2+1}$. Thus, be careful when solving the differential equation to consider all possibilities. Since the solution $y = 0$ cannot be divided by in the step immediately afterwards, it must be considered separately.

1.3 Linear Inhomogeneous

Appears as: $y' = a(x)y + f(x)$

1.3.1 Integrating Factor Method

If $u = e^{-\int a dx}$, then $u' = -ae^{-\int a dx} = -ua$

$y' = ay + f$
$y' - ay = f$
$uy' - uay = uf$
$uy' + u'y = uf$
$(uy)' = uf$
$uy = \int uf dx + c$
$y = \frac{1}{u}(\int uf dx + c)$

Ex: $y' = y\cos(x) + \cos(x)$
This is the form of the solution that we want, so we let the integrating factor be
$u = e^{-\int \cos(x))\, dx} = e^{-\sin(x)}$
$y' - y\cos(x) = cos(x)$
$(y' - y\cos(x))e^{-\sin(x)} = \cos(x)e^{-\sin(x)}$
By product rule, $(y' - y\cos(x))e^{-\sin(x)} = (ye^{-\sin(x)})'$, so
$(ye^{-\sin(x)})' = \cos(x)e^{-\sin(x)}$
Then, a general solution may be found by simply integrating both sides.
$\int (ye^{-\sin(x)})'\, dx = \int \cos(x)e^{-\sin(x)}\, dx$
$ye^{-\sin(x)} = -e^{-\sin(x)} + C$
$y = -1 + Ce^{\sin(x)}$

Actually, this particular differential equation can also be solved using separation of variables in the following manner.
$y' = y\cos(x) + \cos(x)$
$y' = (y+1)\cos(x)$
$\frac{dy}{y+1} = \cos(x)\, dx$
$\int \frac{dy}{y+1} = \int \cos(x)\, dx$
$ln|y+1| = \sin(x) + C$
$y = -1 + Ce^{\sin(x)}$
So, basically, use the method you think is more applicable to the situation, because either will give the same answer if the criteria for the use of both is met.

1.3.2 Variation of Parameters

Consider a solution to the homogeneous part $y_h = e^{\int a dx}$
Then $y_h' = ae^{\int a dx} = ay_h$ and let $y = vy_h$

$y' = ay + f$
$(vy_h)' = a(vy_h) + f$
$vy_h' + v'y_h = vy_h' + f$

$v'y_h = f$
$v' = \frac{f}{y_h}$
$v = \int \frac{f}{y_h} dx + c$
$y = vy_h$

This is really the same method as integrating factors as the integrating factor found before solves the homogeneous solution. However, it is worth seeing it in this light with a different example.

Ex: $y' = \frac{y}{x} + \ln(x)$
The homogeneous equation would then be
$y' = \frac{y}{x}$
This can be solved using separation of variables as follows.
$\frac{dy}{y} = \frac{dx}{x}$
$\int \frac{dy}{y} = \int \frac{dx}{x}$
$\ln|y| = \ln|x| + C$
$y = Cx$
Now, we allow the constant itself to vary with x and look for a solution of the form
$y = xu(x)$
Plugging into the differential equation,
$y' = u(x) + xu'(x)$
$u(x) + xu'(x) = \frac{xu(x)}{x} + \ln(x)$
$xu'(x) = \ln(x)$
$u'(x) = \frac{\ln(x)}{x}$
$u = \int \frac{\ln(x)}{x} dx$
Hopefully you remember how to do this integral! Let $v = \ln(x)$. Then, $dv = \frac{1}{x} dx$, so
$u = \int \frac{\ln(x)}{x} dx = \int v\, dv = \frac{v^2}{2} + C = \frac{(\ln(x))^2}{2} + C$
Thus, the entire solution is
$y = xu(x) = x(\frac{(\ln(x))^2}{2} + C) = \frac{x(\ln(x))^2}{2} + Cx$
which is also of course entirely attainable using the method of integrating factors as well.

1.4 Exact

The function $F(x,y) = c$ is exact if $\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$
The equation in the form of $P(x,y)dx + Q(x,y)dy = 0$ is exact if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

If the equation is exact, integrate Pdx and Qdy simultaneously to obtain an implicit solution
$F(x,y) = c = \int Pdx = \int Qdy$

Ex: $\frac{dy}{dx} = \frac{x^2 - 2\frac{y}{x}}{\ln(x^2)}$
To check that this equation is in fact exact, we have to write it in the form above.
$(x^2 - 2\frac{y}{x})\, dx - \ln(x^2)\, dy = 0$
Then we can check if the mixed partial derivatives are equal.
$\frac{\partial(x^2 - 2\frac{y}{x})}{\partial y} = -\frac{2}{x} = \frac{\partial(-\ln(x^2))}{\partial x}$
Therefore, we can try finding the solution. (Don't forget $\ln(x^2) = 2\ln(x)$)
$F(x,y) = \int (x^2 - 2\frac{y}{x})\, dx = \frac{x^3}{3} - 2y\ln(x) + g(y)$
The function g(y) is necessary as any function of only y will disappear when differentiated with respect to x.
We can now take $\frac{\partial F(x,y)}{\partial y}$ and set it equal to $-\ln(x^2)$ to see what g(y) must be.
$\frac{\partial F(x,y)}{\partial y} = -2\ln(x) + g'(y) = -\ln x^2 + g'(y)$ which is equal to $-\ln(x^2)$
$g'(y) = 0$
$g(y) = C$

$F(x,y) = \frac{x^3}{3} - 2y\ln(x) + g(y) = \frac{x^3}{3} - 2y\ln(x) + C_1 = C_2$
Again, the constants are arbitrary and so is the difference, so we can really just combine them.
$F(x,y) = \frac{x^3}{3} - 2y\ln(x) = C$
For exact differentials, they are usually not separable, however, here, we end up being lucky enough to find an explicit solution.
$\frac{x^3}{3} - 2y\ln(x) = C$
$\frac{x^3}{3} - C = 2y\ln(x)$
$y = \frac{x^3-3C}{6\ln(x)}$
Again, an arbitrary constant -3C; let us just call it C.
$y = \frac{x^3+C}{6\ln(x)}$ or $y = \frac{x^3+C}{\ln(x^6)}$

It may not be so obvious at first, but the differential equation above can also be solved using integrating factors or variation of parameters (as you should verify).

1.4.1 Exact with Integrating Factor

If the function is not exact, multiplying by an integrating factor may make it exact.
For the function $uP(x,y)dx + uQ(x,y)dy = 0$ to be exact, $\frac{\partial uP}{\partial y} = \frac{\partial uQ}{\partial x}$
The integrating factor can be found if it is only a function of x or y alone.
If the integrating factor is a function of x only, let $u(x) = e^{\int h dx}$ where $h = \frac{1}{Q}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$
If the integrating factor is a function of y only, let $u(y) = e^{-\int g dy}$ where $g = \frac{1}{P}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

Ex: $\frac{dy}{dx} = \frac{xy}{y-x^2-y^2}$
Ok, let's see if this happens to be exact (even though it would defeat the purpose if it was).
$xy\,dx + (x^2+y^2-y)\,dy = 0$
$\frac{\partial(x^2+y^2-y)}{\partial x} = 2x \neq x = \frac{\partial(xy)}{\partial y}$
So, we do not have an exact differential. However, we do have that
$\frac{\partial(xy)}{\partial y} - \frac{\partial(x^2+y^2-y)}{\partial x} = x - 2x = -x$
Thus, $g = \frac{1}{P}(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}) = \frac{-x}{xy} = -\frac{1}{y}$
This satisfies the conditions above, where g must be a function of y alone. Therefore, we should look for an integrating factor of the following form.
$u(y) = e^{-\int -\frac{1}{y}\,dy} = e^{ln|y|+C}$ It is less obvious here, but the constant is also unnecessary as it will be multiplied to both sides of the inexact differential equation, and so from here on, the constant will be omitted.
$u(y) = e^{ln|y|} = y$
Thus, the inexact equation $xy\,dx + (x^2+y^2-y)\,dy = 0$ becomes $xy^2\,dx + (x^2y+y^3-y^2)\,dy = 0$ We can now apply the methods of the previous section to check that this is exact and solve it.
$\frac{\partial(xy^2)}{\partial y} = 2xy = \frac{\partial(x^2y+y^3-y^2)}{\partial x}$
Therefore, the equation is now exact.
$F(x,y) = \frac{x^2y^2}{2} + g(y)$
$\frac{\partial F(x,y)}{\partial y} = \frac{\partial(\frac{x^2y^2}{2}+g(y))}{\partial y} = x^2y + g'(y) = x^2y + y^3 - y^2$
$g'(y) = y^3 - y^2$
$g = \frac{y^4}{4} - \frac{y^3}{3} + C$
$F(x,y) = \frac{x^2y^2}{2} + \frac{y^4}{4} - \frac{y^3}{3} = C$
This time, it is not possible to separate, so it is the simplest form we can leave the solution in.

1.5 Homogeneous

For the function $P(x,y)dx + Q(x,y)dy = 0$
If $P(tx,ty) = t^kP(x,y)$ and $Q(tx,ty) = t^kQ(x,y)$
Then let $y = xv$ and $dy = vdx + xdv$

$P(x,y)dx + Q(x,y)dy = 0$
$P(x,xv)dx + Q(x,xv)(vdx + xdv) = 0$
Then separate and integrate to solve.

Ex: $\frac{dy}{dx} = \frac{x+y}{x-y}$
We can once again split this up into the form above and check whether or not the equation is exact.
$(x+y)\,dx + (y-x)\,dy = 0$
$\frac{\partial(x+y)}{\partial y} = 1 \neq -1 = \frac{\partial(y-x)}{\partial x}$
So, the equation is not exact. However, let us check that it satisfies the conditions above.
$P(x,y) = x+y \Rightarrow P(tx,ty) = (tx)+(ty) = t(x+y) = tP(x,y)$ $Q(x,y) = y-x \Rightarrow Q(tx,ty) = (ty)-(tx) = t(y-x) = tQ(x,y)$
Therefore, the equations are in fact homogeneous, so we can try the substitution above.
$y = xv$
$(x+xv)\,dx + (xv-x)\,(xdv+vdx) = 0$
$(x+xv-xv+xv^2)\,dx + (x^2v-x^2)\,dv = 0$
$x(1+v^2)\,dx + x^2(v-1)\,dv = 0$
Now, we can separate the solution and solve for what v should be.
$x(1+v^2)\,dx = x^2(1-v)\,dv$
$\frac{1-v}{1+v^2}\,dv = \frac{dx}{x}$
Then, we can integrate and find an implicit solution for v.
$\int \frac{1-v}{1+v^2}\,dv = \int \frac{dx}{x}$
$\tan^{-1}(v) - \frac{1}{2}\ln(1+v^2) = \ln(x) + C$
We probably cannot invert this to solve for v, but we can plug in the expression $v = \frac{y}{x}$ to get an expression in terms of the original variables.
$\tan^{-1}(y/x) - \frac{1}{2}\ln(\frac{x^2+y^2}{x^2}) = \ln(x) + C$

2 Existence, Uniqueness, Autonomy, Stability

Existence: Consider the normal form $y' = f(x,y)$. Restrict the interval of existence according to the continuity of both f and the solution.

Uniqueness (Cauchy Theorem): Consider the normal form $y' = f(x,y)$. If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are both continuous on the rectangle R in the xy-plane, then a solution to the initial value problem is unique so long as $(x,y) \in R$.

Autonomy: An autonomous equation is of the form $y' = f(y)$ and a non-autonomous equation is of the form $y' = f(x,y)$. The solution to an autonomous equation may be shifted left or right.

Stability (autonomous equations): Consider the normal form $y' = f(y)$. The points at which $y' = 0$ are called equilibrium points (or fixed points), and they represent constant equilibrium solutions.

A equilibrium point is stable if a solution curve starting near it approaches it as x approaches $+\infty$ ($f'(y_i) < 0$)

A equilibrium point is unstable if a solution curve starting near it moves away it as x approaches $+\infty$ ($f'(y_i) > 0$)

3 Second Order Differential Equations

3.1 Linear Homogeneous with Constant Coefficients

For the form of $y'' + py' + qy = 0$
Assume that $y = e^{\lambda t}$ is a solution
Then $y'' = \lambda^2 e^{\lambda t}$

Plugging this back into the second order differential equation, the solution is $y = e^{\lambda t}$ where $\lambda = \frac{-p \pm \sqrt{p^2-4q}}{2}$
If the solution is imaginary, use Euler's identity ($e^{i\theta} = \cos(\theta) + i\sin(\theta)$)

Ex: Case 1: 2 real roots
$y'' - 2y' - 3y = 0$
Let us try the solution $e^{\lambda x}$.
$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} 23 e^{\lambda x} = 0$
Since exponentials are never 0, we know $\lambda^2 - 2\lambda 2 - 3 = 0$.
This polynomial can be factored (though in some rare cases, quadratic formula must be used) to get the roots.
$(\lambda + 1)(\lambda - 3) = 0$
$\lambda = -1, 3$
The solution then takes the form of a linear combination of the exponentials.
$y = C_1 e^{-x} + C_2 x e^{3x}$.

Ex: Case 2: Repeated Roots
$y'' + 2y' + y = 0$
Again, let us try the solution $e^{\lambda x}$.
$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + e^{\lambda x} = 0$
Since exponentials are never 0, we know $\lambda^2 + 2\lambda + 1 = 0$.
This quadratic can be easily factored (as most with repeated roots can be) into $(\lambda + 1)^2 = 0$. Therefore, the double root is $\lambda = -1$, which gives a single solution e^{-x}. There are various ways to get the other solution such as reduction of order or method of operators, but it is usually easiest and most time efficient to remember that the other independent solution is x multiplied by the first solution, in this case xe^{-x}.
Therefore, the solution is
$y = C_1 e^{-x} + C_2 x e^{-x}$.

Ex: Case 3: Complex Roots
$y'' + 3y' + 5y = 0$
Let us try the solution $e^{\lambda x}$.
$\lambda^2 e^{\lambda x} + 3\lambda e^{\lambda x} + 5e^{\lambda x} = 0$
Since exponentials are never 0, we know $\lambda^2 + 3\lambda + 5 = 0$. By quadratic formula,
$\lambda = \frac{-3 \pm \sqrt{3^2 - (4)(5)}}{2}$
$\lambda = \frac{-3}{2} + \frac{i\sqrt{11}}{2}$
$e^{(x)(\frac{-3}{2} + \frac{i\sqrt{11}}{2})} = e^{\frac{-3x}{2}}(\cos\left(\frac{x\sqrt{11}}{2}\right) + i\sin\left(\frac{x\sqrt{11}}{2}\right))$
Both the sine and the cosine solutions form independent solutions to the differential equation, so we have
$y = C_1 e^{\frac{-3x}{2}} \cos\left(\frac{x\sqrt{11}}{2}\right) + C_2 e^{\frac{-3x}{2}} \sin\left(\frac{x\sqrt{11}}{2}\right)$

Basically, plug in $e^{\lambda x}$ to start, get the polynomial in terms of λ, find the roots, determine which case you are in, and apply the tactics of the case for your problem (in that order).

3.2 Linear Inhomogeneous with Constant Coefficients

For the form of $y'' + py' + qy = g(x)$
The solution is simply the homogeneous solution plus the particular solution.
$y = y_h + y_p$

3.3 Linear Inhomogeneous with Exponential Forcing Function

Form: $y'' + py' + qy = e^{bx}$
Particular Solution: $y_p = ce^{bx}$

We will examine two differential equations with exponential forcing terms. The "usual" case, where the method above works and the "rare" case when the forcing term is part of the homogeneous solution.

Case 1: "Usual" Case
$y'' + 4y' + 4y = 3e^{3x}$
First, we have to solve the homogeneous solution, or $y'' + 4y' + 4y = 0$.
The fundamental polynomial in this case is $\lambda^2 + 4\lambda + 4 = 0$, which factors into $(\lambda + 2)^2 = 0$. Therefore, the homogeneous solution is
$y_h = C_1e^{-2x} + C_2xe^{-2x}$
Now, we guess that the particular solution will be of the form $y_p = Ae^{3x}$. Let us plug this into the inhomogeneous differential equation to solve for the constant A.
$y_p{}' = 3Ae^{3x}$
$y_p{}' = 9Ae^{3x}$
$(9Ae^{3x}) + 4(3Ae^{3x}) + 4(Ae^{3x}) = 3e^{3x}$
$25Ae^{3x} = 3e^{3x}$
Therefore, for the last equation to make sense, the constant A must be $\frac{3}{25}$. So, the total solution is
$y = C_1e^{-2x} + C_2xe^{-2x} + \frac{3}{25}e^{3x}$
An important thing to keep in mind is that particular solutions are not necessarily unique; it is just when guessing a particular solution of the form $y_p = Ae^{3x}$ that the particular solution is unique.

Case 2: "Rare" Case
$y'' + 4y' + 4y = -3e^{-2x}$
From the previous case, we know the homogeneous solution, since the homogeneous part is the same, is $y_h = C_1e^{-2x} + C_2xe^{-2x}$.
The problem is that when we plug in $y_p = Ae^{-2x}$, we instead get 0, since this guess is part of the homogeneous solution. The trick is to notice that the differential equation can instead by written like this.
$(\frac{d}{dx} + 1)(\frac{d}{dx} + 1)y = -3e^{-2x}$
We have decomposed the differential equation into a series of operators. We tried $y_p = Ae^{-2x}$ and this gave us 0. If we try one power higher or $y_p = Axe^{-2x} + Be^{-2x}$, this will also be in the homogeneous solution since the double operator $(\frac{d}{dx} + 1)$ implies that it will send solutions up to Axe^{-2x} to 0.
However, trying the solution $y_p = Ax^2e^{-2x} + Bxe^{-2x} + Ce^{-2x}$ will not get sent to 0 by the operator; the equation will get reduced down to simply some multiple of e^{-2x}, but this is what we want. Actually, since $Bxe^{-2x} + Ce^{-2x}$ are part of the homogeneous solution, we need only try $y_p = Ax^2e^{-2x}$.
$y_p{}' = -2Ax^2e^{-2x} + 2Axe^{-2x} = (-2x^2 + 2x)Ae^{-2x}$
$y_p{}'' = -2(-2x^2 + 2x)e^{-2x} + (-4x + 2)Ae^{-2x} = (4x^2 - 8x + 2)Ae^{-2x}$
Now, we plug in the derivatives into the differential equation.
$(4x^2 - 8x + 2)Ae^{-2x} + 4(-2x^2 + 2x)Ae^{-2x} + 4Ax^2e^{-2x} = -3e^{-2x}$
$2Ae^{-2x} = -3e^{-2x}$
As expected, all x and x^2 terms disappeared, allowing us to solve for the constant A as usual.
$A = \frac{-3}{2}$
Therefore, the solution is
$y = C_1e^{-2x} + C_2xe^{-2x} - \frac{3}{2}x^2e^{-2x}$
Just a word of caution: If you attempt to try this method for equations such as $y'' + 4y' + 4y = -3xe^{-2x}$,

you do indeed need to go up to the third power x forcing term, or $y_p = Ax^3e^{-2x} + Be^{-2x}$ as the differential equation will reduce the powers of x twice (as seen in the last example). So, you should always find the homogeneous solution and determine how many powers of x are being lost through putting it through the differential equation.

3.4 Linear Inhomogeneous with Trigonometric Forcing Function

Form: $y'' + py' + qy = A\cos(\omega x) + B\sin(\omega x)$
Particular Solution: $y_p = c_1\cos(\omega x) + c_2\sin(\omega x) = ce^{i\omega x}$

Ex: Case 1: "Usual Case"
$y'' + 2y' + 2y = cos(x) + 2sin(x)$
First, we have to solve the homogeneous equation.
$y'' + 2y' + 2y = 0$
The polynomial associated with this equation is the following.
$\lambda^2 + 2\lambda + 2 = 0$
This gives, through quadratic formula, the roots
$\lambda = \frac{-2\pm\sqrt{2^2-4(2)}}{2} = -1 \pm i$
Therefore, the homogeneous solution is $y_h = C_1e^{-x}\cos(x) + C_2e^{-x}\sin(x)$. You should observe that the forcing terms are not in this solution.
Now, we can guess what the particular solution is by using the formula above, $y_p = A\cos(x) + B\sin(x)$. We need to solve for both A and B.
$y_p' = -A\sin(x) + B\cos(x)$
$y_p'' = -A\cos(x) - B\sin(x)$
We can plug in all the terms into the differential equation now and solve.
$-A\cos(x) - B\sin(x) + 2(-A\sin(x) + B\cos(x)) + 2(A\cos(x) + B\sin(x)) = cos(x) + 2sin(x)$
$(-A + 2B + 2A)\cos(x) + (-B - 2A + 2B)\sin(x) = \cos(x) + 2\sin(x)$
$(A + 2B)\cos(x) + (B - 2A)\sin(x) = \cos(x) + 2\sin(x)$
This gives us a system of equations to solve for A and B. Feel free to use whatever method you feel most comfortable with to solve it; we will use substitution.
$A + 2B = 1$
$B - 2A = 2$

From the second equation, $B = 2A + 2$.
$A + 2B = A + 2(2A + 2) = 1 \Rightarrow 5A + 4 = 1 \Rightarrow A = \frac{-3}{5}$
From this, we also get that $B = 2A + 2 = 2(\frac{-3}{5}) + 2 = \frac{4}{5}$.
Therefore, the particular solution is $y_p = \frac{-3}{5}\cos(x) + \frac{4}{5}\sin(x)$ and the general solution is
$y = C_1e^{-x}\cos(x) + C_2e^{-x}\sin(x) - \frac{3}{5}\cos(x) + \frac{4}{5}\sin(x)$

Ex: Case 2: "Rare" Case
$y'' + 2y' + 2y = e^{-x}\cos(x)$
As in the previous example, the homogeneous solution is given by $y_h = C_1e^{-x}\cos(x)+C_2e^{-x}\sin(x)$. However, this time, the forcing term is part of the homogeneous solution, meaning we cannot simply guess a constant multiple of it. Once again, we look at the operator that make up the differential equation.
$(\frac{d^2}{dx^2} + 2\frac{d}{dx} + 2)y = e^{-x}\cos(x)$
We cannot simplify the operator on the left anymore than the form it is written in, but this is okay as we know that the forcing term solves the homogeneous equation only once and so all powers of x will be reduced once. Therefore, we should try $y_p = Axe^{-x}\cos(x) + Bxe^{-x}\sin(x)$ or more compactly, $y_p = (Ax\cos(x) + Bx\sin(x))e^{-x}$
$y_p' = (A\cos(x) - Ax\sin(x) + B\sin(x) + Bx\cos(x))e^{-x} - (Ax\cos(x) + Bx\sin(x))e^{-x} =$
$((B - A)x\cos(x) + (-B - A)x\sin(x) + A\cos(x) + B\sin(x))e^{-x}$
$y_p'' = (-(B-A)x\sin(x) + (B-A)\cos(x) + (-B-A)x\cos(x) + (-B-A)\sin(x) - A\sin(x) + B\cos(x))e^{-x} -$

$((B-A)x\cos(x)+(-B-A)x\sin(x)+A\cos(x)+B\sin(x))e^{-x} = (-2Bx\cos(x)+2Ax\sin(x)-(2A+2B)\sin(x)+(2B-2A)\cos(x))e^{-x}$
Finally, we can enter in the derivatives into the differential equation and solve for A and B.
$(-2Bx\cos(x) + 2Ax\sin(x) - (2A+2B)\sin(x) + (2B-2A)\cos(x))e^{-x}$
$(2(B-A)x\cos(x) + 2(-B-A)x\sin(x) + 2A\cos(x) + 2B\sin(x))e^{-x}$
$(2Ax\cos(x) + 2Bx\sin(x))e^{-x}$

$(-2A\sin(x) + 2B\cos(x))e^{-x} = e^{-x}\cos(x)$
With so many terms, it is probably easier to add vertically. You should use the fact that the terms multiplied by x cancel out to ensure your work is correct (though it comes a little late). Also, we have $A = 0, B = \frac{1}{2}$.
Therefore, the general solution is
$y = C_1 e^{-x}\cos(x) + C_2 e^{-x}\sin(x) + \frac{1}{2}xe^{-x}\sin(x)$

3.5 Linear Inhomogeneous with Polynomial Forcing Function

Form: $y'' + py' + qy = f(x)$
Particular Solution: $y_p = xP(x)$

Ex: $y'' + 5y' + 4y = 3x^2$
We must first solve the homogeneous solution, $y'' + 5y' + 4y = 3x^2$.
The fundamental polynomial in this case is $\lambda^2 + 5\lambda + 4 = 0$, which factors into $(\lambda+4)(\lambda+1) = 0$. The roots are $\lambda = -1, -4$ and so we have that the homogeneous solution is
$y_h = C_1 e^{-x} + C_2 e^{-4x}$
Now, we guess that the particular solution is of the form
$y_p = Ax^2 + Bx + C$
Note that all of the terms must be kept in this case since none of them are part of the homogeneous solution.
$y_p' = 2Ax + B$
$y_p'' = 2A$
Therefore, upon plugging in the derivatives into the differential equation, we get the following.
$2A + 5(2Ax + B) + 4(Ax^2 + Bx + C) = 3x^2$
$4Ax^2 + (10A + 4B)x + (2A + C) = 3x^2$
This gives the 3x3 system of equations
$4A = 3$
$10A + 4B = 0$
$2A + C = 0$
From the first equation, we get $A = \frac{3}{4}$. From the second equation, applying $A = \frac{3}{4}$, we get $B = \frac{-15}{8}$. From the third equation, applying $A = \frac{3}{4}$, we get $C = \frac{-3}{2}$. Therefore, the particular solution is $y_p = \frac{3}{4}x^2 + \frac{-15}{8}x + \frac{-3}{2}$ and the general solution is
$y = C_1 e^{-x} + C_2 e^{-4x} + \frac{3}{4}x^2 - \frac{15}{8}x - \frac{3}{2}$

You will not really have to worry about polynomial terms being part of the homogeneous solution. The only 2 cases when it can happen are when y is missing from the left side, or both y and y'are missing. In the first case, you have something that looks like this.
$y'' + Ay' = f(x)$
where A is a constant and f(x) is a polynomial. It is true that all constant polynomial disappear when put through the left side, however, you can also simply integrate the equation to get
$y' + Ay = \int f(x)\,dx + C$
which is now first order (actually this can be applied for f(x) non-polynomial, so remember this trick). In the case when both y and y'are missing, you end up with
$y'' = f(x)$
which you should simply integrate twice, regardless of what f is.

Also, a general note on forcing terms. If you have a differential equation of the form

$y'' + 2y' + y = x^2 + e^x$
and you happen to know that f(x) is the particular term for
$y'' + 2y' + y = x^2$
and similarly g(x) is the particular solution for
$y'' + 2y' + y = e^x$
Then, f(x) + g(x) is the forcing term for
$y'' + 2y' + y = x^2 + e^x$
This should help ease calculations as you need only solve for one particular solution at a time ($A\cos(x) + B\sin(x)$ and all such trigonometric functions with the same argument count as one forcing term, so you should do them together to save doing the same particular solution twice).

3.6 Variation of Parameters

If you do not have a linear differential equation or you have an integrating factor that does not fall into the above categories of exponentials, trigonometric functions, or polynomials, you have to get a method called the variation of parameters. The general idea is as follows.
Let us say you have a differential equation of this form.
$y'' + p(x)y' + q(x)y = f(x)$
Here, p(x) and q(x) need not be constants. Let us also assume you happen to know the solution to the homogeneous equation, $y'' + p(x)y' + q(x)y = 0$, which is tricky to find and the hard part about using this method, so usually the homogeneous solution is given or the equation is linear.
However, if you do happen to have 2 linearly independent solutions, $\phi_1(x)$, $\phi_2(x)$, then the particular solution is of the form
$y_p = u_1\phi_1 + u_2\phi_2$
where u_1 and u_2 are given by the formulas
$u_1 = -\int \frac{\phi_2(x)f(x)}{\phi_1(x)\phi_2'(x)-\phi_1'(x)\phi_2(x)}\,dx$
$u_2 = \int \frac{\phi_1(x)f(x)}{\phi_1(x)\phi_2'(x)-\phi_1'(x)\phi_2(x)}\,dx$
The denominator in the integral is often expressed differently, as the Wronskian determinant.

$$W = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1(x)\phi_2'(x) - \phi_1'(x)\phi_2(x)$$

In this new notation, the integrals for u_1 and u_2 are written as follows.
$u_1 = -\int \frac{\phi_2(x)f(x)}{W(x)}\,dx$
$u_2 = \int \frac{\phi_1(x)f(x)}{W(x)}\,dx$

Ex: $xy'' + 2y' + xy = x$
This is definitely not a form we have seen before, so we need to be given the two linearly independent solutions. Well, here they are.
$\phi_1 = \frac{\cos(x)}{x}$
$\phi_2 = \frac{\sin(x)}{x}$
The first step in these sorts of problems is verifying these are in fact solutions. $\phi_1' = \frac{-x\sin(x)-\cos(x)}{x^2}$
$\phi_1'' = \frac{-x^2\cos(x)+2x\sin(x)+2\cos(x)}{x^3}$
$x(\frac{-x^2\cos(x)+2x\sin(x)+2\cos(x)}{x^3}) + 2(\frac{-x\sin(x)-\cos(x)}{x^2}) + x(\frac{\cos(x)}{x}) = 0$
$\phi_2' = \frac{x\cos(x)-\sin(x)}{x^2}$
$\phi_2'' = \frac{-x^2\sin(x)+2x\cos(x)-2\sin(x)}{x^3}$
$x(\frac{-x^2\sin(x)+2x\cos(x)-2\sin(x)}{x^3}) + 2(\frac{x\cos(x)-\sin(x)}{x^2}) + x(\frac{\sin(x)}{x}) = 0$

Now that we have confirmed that our two functions are linear independent homogeneous solutions, we can apply the formulas above. First, we have to calculate the Wronskian.
$W = \phi_1(x)\phi_2'(x) - \phi_1'(x)\phi_2(x) = (\frac{\cos(x)}{x})(\frac{x\cos(x)-\sin(x)}{x^2}) - (\frac{\sin(x)}{x})(\frac{-x\sin(x)-\cos(x)}{x^2}) = \frac{x(\sin^2(x^2)+\cos(x^2))}{x^3} = \frac{1}{x^2}$

Now, we have to be careful applying the formulas above, as they were written for equations in the form $y'' + p(x)y' + q(x)y = f(x)$, however, our equation is in the form $xy'' + 2y' + xy = x$. We can remedy this by dividing both sides of the equation by x.
$y'' + \frac{2}{x}y' + y = 1$
So, really, our forcing term is $f(x) = 1$, not x. $u_1 = -\int \frac{(1)\frac{\sin(x)}{x}}{\frac{1}{x^2}}\,dx = -\int x\sin(x)\,dx$
This last integral must be done by parts. Let $du = -\sin(x)dx$, $v = x$. Then, $u = cos(x)$, $dv = dx$, so
$u_1 = -\int x\sin(x)\,dx = x\cos(x) - \int \cos(x)\,dx = x\cos(x) - \sin(x) + C_1$
$u_2 = \int \frac{(1)\frac{\cos(x)}{x}}{\frac{1}{x^2}}\,dx = \int x\cos(x)\,dx$
Let $du = \cos(x)dx$, $v = x$. Then, $u = sin(x)$, $dv = dx$, so
$u_2 = \int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x) + C_2$
Therefore, the general solution is given by the following.
$y = (x\cos(x) - \sin(x) + C_1)(\frac{\cos(x)}{x}) + (x\sin(x) + \cos(x) + C_2)(\frac{\sin(x)}{x}) = 1 + C_1\frac{\sin(x)}{x} + C_2\frac{\cos(x)}{x}$
It is not hard to see that this indeed fits the differential equation after showing that $\frac{\cos(x)}{x}$ and $\frac{\sin(x)}{x}$ are part of the homogeneous solution.

3.7 Reduction of Order

This is another way of solving for the general solution of a differential equation, however, you are only given one solution to the homogeneous equation. If you are given 2 linearly independent solutions, you should definitely instead use Variation of Parameters as the integrals are much easier to solve that the proceeding formulation.
Form: $y'' + p(x)y' + q(x)y = f(x)$
One solution to homogeneous equation: $\phi(x)$
General solution: $y = u(x)\phi(x)$
We are not lucky enough to have a general formula in this case, so the best we can do is plug in y into the differential equation and solve for u(x).
$y' = u'(x)\phi(x) + u(x)\phi'(x)$
$y'' = u''(x)\phi(x) + 2u'(x)\phi'(x) + u(x)\phi''(x)$
$(u''(x)\phi(x) + 2u'(x)\phi'(x) + u(x)\phi''(x)) + p(x)(u'(x)\phi(x) + u(x)\phi'(x)) + q(x)(u(x)\phi(x)) = f(x)$
Now, we regroup the terms in terms of derivatives of u.
$(u''(x))(\phi(x)) + (u'(x))(2\phi'(x) + p(x)\phi(x)) + (u(x))(\phi''(x) + p(x)\phi'(x) + q(x)\phi(x)) = f(x)$
However, we know that the $\phi(x)$ is part of the homogeneous solution, so the last term $(u(x))(\phi''(x) + p(x)\phi'(x) + q(x)\phi(x))$ must be 0.
This leaves a differential equation for u(x) without u(x).
$(u''(x))(\phi(x)) + (u'(x))(2\phi'(x) + p(x)\phi(x))) = f(x)$
We will solve this by making the substitution $v = u'$ $(v'(x))(\phi(x)) + (v(x))(2\phi'(x) + p(x)\phi(x))) = f(x)$
This leaves a first order equation for v(x) that can then be solved for.

Ex: Proof of repeated second order equation
$y'' - 2ay' + a^2y = 0$, a is a real number
We already know one solution to this equation is e^{ax}.
We can now apply the technique of reduction of order above.
$y = u(x)e^{ax}$ $y' = u'(x)e^{ax} + au(x)e^{ax} = e^{ax}(u'(x) + au(x))$
$y'' = ae^{ax}(u'(x) + au(x)) + e^{ax}(u''(x) + au'(x)) = e^{ax}(u''(x) + 2au'(x) + a^2u(x))$
$e^{ax}(u''(x) + 2au'(x) + a^2u(x)) - 2a(e^{ax}(u'(x) + au(x))) + a^2u(x)e^{ax} = 0$
$e^{ax}(u''(x)) = 0$
Since we know that exponentials are never 0, $u''(x) = 0$
This gives rise to the solution $u(x) = C_1x + C_2$, which gives
$y = (C_1x + C_2)e^{ax} = C_1e^{ax} + C_2xe^{ax}$

Ex: $x^2y'' - 2y = x^3e^x$
This differential equation has a homogeneous solution that can be found through guessing of $\phi(x) = \frac{1}{x}$.
Now that we have one homogeneous solution, let us find the general solution.
$y = \frac{u(x)}{x}$
$y' = \frac{xu'-u}{x^2}$
$y'' = \frac{x^2u''-2xu'+2u}{x^3}$
$(x^2)(\frac{x^2u''-2xu'+2u}{x^3}) - 2(\frac{u(x)}{x}) = x^3e^x$
$xu'' - 2u' = x^3e^x$
Now, we can make the substitution $v = u'$ and solve for v.
$xv' - 2v = x^3e^x$
This can be solved using an integrating factor.
$v' - \frac{2}{x}v = x^2e^x$
$\mu = e^{-\int \frac{2}{x}\,dx} = e^{-2\ln|x|} = \frac{1}{x^2}$
$\frac{v}{x^2} = \int e^x\,dx$
$\frac{v}{x^2} = e^x + C_1$
$v = x^2e^x + C_1x^2$
Now, we remember that $v = u'$ and substitute this equation back in.
$u' = x^2e^x + C_1x^2$
The second term is easy to integrate; the first one must be done by parts. Let $w = x^2$,$dp = e^xdx$. Then, $dw = 2x\,dx$, $p = e^x$. $u = \int x^2e^x + C_1x^2\,dx = C_1x^3 + x^2e^x - \int 2xe^x\,dx$
Again, let $w = -2x$, $dp = e^x\,dx$. Then, $dw = -2\,dx$, $p = e^x$.
$u = C_1x^3 + x^2e^x - 2xe^x - \int e^x(-2)\,dx = C_1x^3 + x^2e^x - 2xe^x + 2e^x + C_2 = C_1x^3 + e^x(x^2 - 2x + 2) + C_2$
Then, the general solution has the form
$y = \frac{u}{x} = C_1x^2 + \frac{C_2}{x} + e^x(x - 2 + \frac{2}{x})$

4 Systems of Linear Differential Equations

4.1 Form of the Problem

Form: $\frac{d\vec{y}}{dx} = A\vec{y}$ where A is an nxn matrix.

Ex:

$$\frac{d\vec{y}}{dx} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{y}$$

$$\frac{d\vec{y}}{dx} = \begin{bmatrix} 3 & 1 & 5 \\ 1 & -4 & 7 \\ -6 & 8 & 9 \end{bmatrix} \vec{y}$$

4.2 Solutions to problems with complete eigenvalues

Although the general solution to problems with incomplete eigenvalues is important, here we will focus on when all eigenvectors of a matrix can be solved for.
Recall that an eigenvector to a matrix solves the equation $\det(A - \lambda I_n) = 0_{nxn}$
Then, the eigenvalues are given found through the kernel of this new matrix, or finding which values of $\vec{x}$ give $(A - \lambda I_n)\vec{x} = \vec{0_n}$. Here, 0_{nxn} denotes the n x n matrix of all zeros and 0_n denotes the vector of size n of all zeros.
Now, suppose that all eigenvalues and eigenvectors corresponding to these eigenvalues can be found and let them be called λ_1, $\vec{v_1}$,...,λ_n, $\vec{v_n}$ where the λs may not be unique, but the eigenvectors must be independent.
Solution: $\vec{y} = C_1e^{\lambda_1 x}\vec{v_1} + ... + C_ne^{\lambda_n x}\vec{v_n}$
It is not hard to see why this linear combination is made up of solutions; simply take the first term and plug it into the differential equation $\frac{d\vec{y}}{dx} = A\vec{y}$. The derivative of $C_1e^{\lambda_1 x}\vec{v_1}$ is simply $C_1\lambda_1e^{\lambda_1 x}\vec{v_1}$ since the exponential appears in every term and contains the only varying part of the expression with respect to x.

Moreover, since $\vec{v_1}$ is an eigenvector of A with eigenvalue v_1, we have $AC_1e^{\lambda_1 x}\vec{v_1} = C_1e^{\lambda_1 x}A\vec{v_1} = C_1e^{\lambda_1 x}\lambda_1\vec{v_1}$. So, we get equality of both sides. By linearity, the argument can be repeated for all terms in the sum. Also, as you might expect, complex eigenvalues lead to trigonometric solutions (example below).

Ex:

$$\frac{d\vec{y}}{dx} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{y}$$

You should work out 2x2 systems before moving onto higher order since 2x2 systems are very fast in solving and so you can focus on concepts instead.
For the example given,

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Now, to find the eigenvalues of the matrix.

$$A - \lambda I_2 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

Since the determinant will only lead to a quadratic in λ, it is safe to simply expand the determinant.
$\det(A - \lambda I_2) = (3-\lambda)(2-\lambda) - (1)(1) = \lambda^2 - 5\lambda + 5$
We need to find out for what values of λ the equation becomes 0, in this case using quadratic formula.
$\lambda^2 - 5\lambda + 5 = 0$
$\lambda = \frac{5 \pm \sqrt{25-(4)(5)}}{2} = \frac{5}{2} \pm \frac{\sqrt{5}}{2}$
Now that we have the eigenvalues, we need to find the eigenvectors. For $\lambda = \frac{5}{2} + \frac{\sqrt{5}}{2}$, we have the following matrix.

$$A - (\frac{5}{2} + \frac{\sqrt{5}}{2})I_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix}$$

Here, it is somewhat difficult to guess what vector when multiplied by the matrix gives the 0 vector. It would also be somewhat tedious to row reduce right away by dividing by $\frac{1-\sqrt{5}}{2}$, so it would instead be better to first swap the rows (which does not change these vectors).

$$\begin{bmatrix} 1 & \frac{-1-\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} & 1 \end{bmatrix}$$

We can now row reduce by subtracting exactly $\frac{1-\sqrt{5}}{2}$ times the first row from the second. Notice that $(\frac{1-\sqrt{5}}{2})(\frac{-1-\sqrt{5}}{2}) = 1$, so both bottom entries become zero after the subtraction.

$$\begin{bmatrix} 1 & \frac{-1-\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix}$$

Now, it is quite easy to quess the form of the eigenvectors. If the second component has some number C_1, the first component must have exactly $\frac{1+\sqrt{5}}{2}$ times it to cancel out in matrix multiplication. Thus, the eigenvectors are

$$C_1 \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

Similarly, for the second eigenvector,

$$A - (\frac{5}{2} - \frac{\sqrt{5}}{2})I_2 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{bmatrix}$$

Swapping the rows gives us the following matrix.

$$\begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & 1 \end{bmatrix}$$

We can see by subtracting $\frac{1+\sqrt{5}}{2}$ times the first row from the second again gives us a row of zeros.

$$\begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix}$$

Therefore, the eigenvectors in this case are

$$C_2 \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

Therefore, the general solution to the system of differential equations is

$$\vec{y} = C_1 \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} e^{(\frac{5}{2}+\frac{\sqrt{5}}{2})x} + C_2 \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} e^{(\frac{5}{2}-\frac{\sqrt{5}}{2})x}$$

Ex:

$$\frac{d\vec{y}}{dx} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{y}$$

You may recognize this as the matrix that rotates vectors by 90°counterclockwise in which case the matrix of course has no eigenvectors since no vector rotated by 90°will be some multiple of itself. If you did not recognize this, the usual analysis works anyways.
Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A - \lambda I_2 = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I_n) = \lambda^2 + 1$$

This quadratic of course has no real roots, but has the complex roots $\pm i$. Now we must find the eigenvector associated with each eigenvalue.
For $\lambda = i$,

$$A - iI_2 = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

Since $i^2 = -1$, the second row is exactly i times the first row. Therefore, after row reducing, we get the following.

$$A - iI_2 = \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix}$$

This gives us the eigenvectors

$$C \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

which would ordinarily give us the general solution

$$C \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{ix}$$

However, this solution is complex, and we are only interested in real solutions. Therefore, we will split the solution into its real and imaginary parts, which must both satisfy the differential equation.

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} e^{ix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} (\cos(x) + i\sin(x)) = \begin{bmatrix} \cos(x) + i\sin(x) \\ -i\cos(x) + \sin(x) \end{bmatrix} = \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix} + i \begin{bmatrix} \sin(x) \\ -\cos(x) \end{bmatrix}$$

Both vectors in the last expression are linearly independent solutions to the differential equation, so we simply take a linear combination of them.

$$\vec{y} = C_1 \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix} + C_2 \begin{bmatrix} \sin(x) \\ -\cos(x) \end{bmatrix}$$

We do not need to examine the eigenvalues for the other root, -i, because the eigenvectors happen to be the same (i.e., in the complex case, you get the second eigenvector for free after finding the first one).

Ex:

$$\frac{d\vec{y}}{dx} = \begin{bmatrix} -3 & 0 & 5 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{bmatrix} \vec{y}$$

The matrices can be made a lot worse than this one for finding eigenvalues, but that is not really the point of 33B.
With what will end up as a cubic polynomial, you can be guaranteed of at least one real root. So, you can try to guess it after expanding out the determinant.

$$\det(A - \lambda I_3) = \begin{vmatrix} -3-\lambda & 0 & 5 \\ 0 & -1-\lambda & 0 \\ -2 & 7 & 3-\lambda \end{vmatrix} = (-3-\lambda) \begin{vmatrix} -1-\lambda & 0 \\ 7 & 3-\lambda \end{vmatrix} + (-2) \begin{vmatrix} 0 & 5 \\ -1-\lambda & 0 \end{vmatrix} =$$

$$\det(A - \lambda I_3) = (-3-\lambda)(-1-\lambda)(3-\lambda) + (-2)(5)(1+\lambda) = (\lambda^2 + 4\lambda + 3)(3-\lambda) - 10(\lambda+1)$$

$$\det(A - \lambda I_3) = (-\lambda^3 - \lambda^2 + 9\lambda + 9) + (-10 - 10\lambda) = -\lambda^3 - \lambda^2 - \lambda - 1$$

Sometimes you get lucky like above and can factor the polynomial directly (though I did choose to ignore it for the sake of practicing synthetic division). We want the determinant to be equal to 0, which is equivalent to finding the roots of the polynomial.

$$-\lambda^3 - \lambda^2 - \lambda - 1 = 0$$

$$\lambda^3 + \lambda^2 + \lambda + 1 = 0$$

Here is where a nice fact from root finding comes into play. If you suspect your polynomial has a rational root of the form $\frac{b}{a}$ where a and b are integers, then a must be a factor of the λ^3 (leading) coefficient and b must be a factor of the ones coefficient. Since both coefficients are 1 in this case, the only possible rational roots are ± 1.

$$\begin{array}{r|rrrr} 1 & 1 & 1 & 1 & 1 \\ & & 1 & 2 & 3 \\ \hline & 1 & 2 & 3 & 4 \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & 1 & 1 & 1 \\ & & -1 & 0 & -1 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

Therefore, the polynomial factors in $(\lambda + 1)(\lambda^2 + 1)$ which has roots $-1, \pm i$ Now, we must find the associated eigenvectors.
For $\lambda = -1$,

$$A + \lambda I_3 = \begin{bmatrix} -2 & 0 & 5 \\ 0 & 0 & 0 \\ -2 & 7 & 4 \end{bmatrix}$$

Subtracting the first row from the third gives the following matrix.

$$\begin{bmatrix} -2 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 7 & -1 \end{bmatrix}$$

Swapping the second and third row gives us the zeros in the bottom of the matrix.

$$\begin{bmatrix} -2 & 0 & 5 \\ 0 & 7 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Again, it is easiest to look at the relations between the columns. The third column is 5/2 of the first column plus $(-\frac{1}{7})$ times the second column or

$$\vec{v_3} = \frac{5}{2}\vec{v_1} - \frac{1}{7}\vec{v_2}$$

Rearranging gives the following.

$$\frac{5}{2}\vec{v_1} - \frac{1}{7}\vec{v_2} - \vec{V_3} = 0$$

So, the vectors we are looking for are

$$C_1 \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{7} \\ -1 \end{bmatrix}$$

Now, we need to find the complex eigenvectors.

$$(A - iI_3) = \begin{bmatrix} -3-i & 0 & 5 \\ 0 & -1-i & 0 \\ -2 & 7 & 3-i \end{bmatrix}$$

To row reduce, we will need to subtract off $\frac{-2}{3+i}$ of the first row from the third.
$\frac{-2}{3+i}\frac{3-i}{3-i} = \frac{-6+2i}{10}$

$$\begin{bmatrix} -3-i & 0 & 5 \\ 0 & -1-i & 0 \\ 0 & 7 & 3-i+5\frac{-6+2i}{10} \end{bmatrix}$$

$$\begin{bmatrix} -3-i & 0 & 5 \\ 0 & -1-i & 0 \\ 0 & 7 & 3-i+(-3+i) \end{bmatrix}$$

$$\begin{bmatrix} -3-i & 0 & 5 \\ 0 & -1-i & 0 \\ 0 & 7 & 0 \end{bmatrix}$$

Basically, we end up with the first row being $\frac{-3-i}{5}$ times the third row, or

$$\vec{v_1} = \frac{-3-i}{5}\vec{v_3}$$

$$\vec{v_1} + \frac{3+i}{5}\vec{v_3}$$

The complex eigenvector is then

$$C \begin{bmatrix} 1 \\ 0 \\ \frac{3+i}{5} \end{bmatrix}$$

We have to split up the product of this vector and the complex exponential to find the real and imaginary components.

$$\begin{bmatrix} 1 \\ 0 \\ \frac{3+i}{5} \end{bmatrix} (\cos(x) + i\sin(x)) = \begin{bmatrix} \cos(x) + i\sin(x) \\ 0 \\ \frac{3\cos(x) - \sin(x) + i(cos(x) + 3\sin(x))}{5} \end{bmatrix} = \begin{bmatrix} \cos(x) \\ 0 \\ \frac{3\cos(x) - \sin(x)}{5} \end{bmatrix} + i \begin{bmatrix} \sin(x) \\ 0 \\ \frac{3\sin(x) + \cos(x)}{5} \end{bmatrix}$$

Therefore, the general solution is

$$\vec{y} = C_1 \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{7} \\ -1 \end{bmatrix} e^{-x} + C_2 \begin{bmatrix} \cos(x) \\ 0 \\ \frac{3\cos(x)-\sin(x)}{5} \end{bmatrix} + C_3 \begin{bmatrix} \sin(x) \\ 0 \\ \frac{3\sin(x)+\cos(x)}{5} \end{bmatrix}$$

4.3 Repeated Eigenvalues, 2x2 systems

Form:

$$\frac{d\vec{y}}{dx} = A\vec{y}$$

where λ is a double root of the characteristic polynomial and $\vec{v}$ is an eigenvector associated with λ.
Solution: $\vec{y} = C_1\vec{v_1}e^{\lambda x} + C_2(\vec{v_2} + x\vec{v_1})e^{\lambda x}$ where $(A - \lambda I_2)\vec{v_2} = \vec{v_1}$
You can verify the second part is a solution by plugging in into the differential equation and rearranging the equation $(A - \lambda I_2)\vec{v_2} = \vec{v_1}$ to get $A\vec{v_2} = \vec{v_1} + \lambda\vec{v_2}$.

Ex:

$$\frac{d\vec{y}}{dx} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \vec{y}$$

$$\det(A - \lambda I_2) = \begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$\det(A - \lambda I_2) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

This quadratic has a double root of $\lambda = 3$. First, we need to find the eigenvector associated with it.

$$A - 3I_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Subtracting the first row from the second gives us the row of zeros in the bottom.

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

The vectors that are sent to the zero vector are $C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Now, we pick a specific vector of this group that is an eigenvector; in this case, it is nice enough to pick $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We want to find a vector $\vec{v_2}$ such that $(A - 3I_2)\vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or,

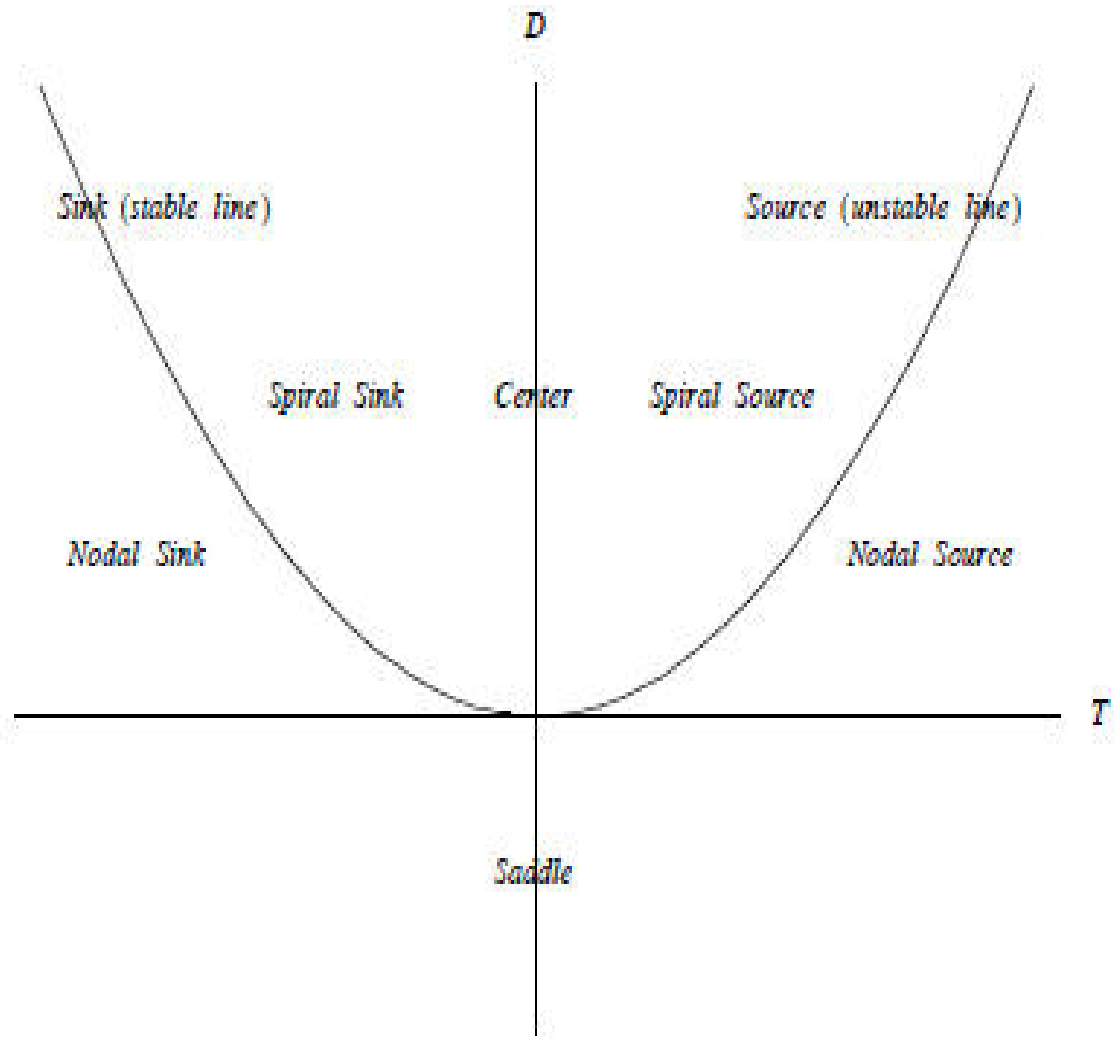
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Here, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ works (and it will almost always be this easy). Therefore, the general solution is the following.

$$\vec{y} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3x} + C_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) e^{3x} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3x} + C_2 \begin{bmatrix} x+1 \\ x \end{bmatrix} e^{3x}$$

4.4 Trace/Determinant Plane

Trace of a matrix: $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a + d$
In general, the trace is the sum of the diagonal entries.



This graph essentially helps break down the many cases for 2x2 matrices. It is useful sometimes to think about how all such matrices vary from one another. However, given a specific problem, the general method is simply to solve it as above and then associate the solution with its characteristics (which avoids having to memorize the chart if you do not want to). The parabolic line has the equation $T^2 = 4D$. Also, the very center corresponds to the point $T = 0, D = 0$, but this leads to the zero matrix, which you should understand regardless.

Let us check the example problems above to see that this graph matches.
Source: As x becomes larger and larger, the solution moves away from the origin in all directions. Sink: As x becomes larger and larger, the solution becomes the origin. Saddle: As x becomes larger, in some directions, $\vec{y}$ moves away from the origin whereas in other directions, $\vec{y}$ moves towards the origin.

Ex:

$$\frac{d\vec{y}}{dx} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \vec{y}$$

This has the solution

$$\vec{y} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3x} + C_2 \begin{bmatrix} x+1 \\ x \end{bmatrix} e^{3x}$$

As x approaches $-\infty$, the solution approaches the origin. However, as x approaches ∞, the second term dominates and the solution goes off to infinity in the direction of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ since this is what $\begin{bmatrix} x+1 \\ x \end{bmatrix}$

approaches for x large. Also, for x approaching $infty$, we also have the second term dominates the first term, and so the solution will once again be parallel to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

. This matches our definition of an unstable line source. Indeed, we could have identified this as a source by noting that $T\left(\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\right) = 9$ and $\det\left(\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\right) = 4 + 2 = 6$. This matches the equation $T^2 = 4D$, for $T > 0$, so the matrix is indeed a line source.

Ex:

$$\frac{d\vec{y}}{dx} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{y}$$

This has the solution

$$\vec{y} = C_1 \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix} + C_2 \begin{bmatrix} \sin(x) \\ -\cos(x) \end{bmatrix}$$

This essentially continues to periodically go around the origin in an ellipse (though never reaching the origin). As such, it is called a "center". Checking with the trace/determinant plane, since D = 1, T = 0, we also see that this matches the graph.

4.5 Exponential of a Matrix and Repeated Eigenvalues

If A is an n x n matrix, $e^A = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ...$
Do not worry about whether this infinite series converges or not – this is not part of the class.
Solution to repeated eigenvalues: If $\vec{v_1}$,...,$\vec{v_n}$ are linearly independent vectors, then the solution is
$\vec{y} = C_1 e^{Ax} \vec{v_1} + ... + C_n e^{Ax} \vec{v_n}$
Any linearly independent vectors will work, but eigenvectors of A, whenever possible, greatly simplify calculations.

Ex:

$$\frac{d\vec{y}}{dx} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & -1 \end{bmatrix} \vec{y}$$

Since we have a lower triangular matrix, we know that the eigenvalues are all -1. However, taking $A + I_3$ gives us the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix} \vec{y}$$

This matrix has 2 linearly independent columns, and so it has only one eigenvector that can be found, $C_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. This leads to the solution $e^{-x} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We must find the other 2 through the rule above.

Note that $e^{Ax}\vec{v} = e^{\lambda x} e^{(A-\lambda I_n)x} \vec{v} = e^{\lambda x}(I_n + xA + \frac{x^2}{2}A^2 + \frac{x^3}{3}A^3 + ...)\vec{v}$, so we need only find vectors that send some power of $A - \lambda I_n$ to zero, as this will send the rest of the terms in the infinite sum to zero as well.

$$(A - \lambda I_3)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

This leads us to another set of vectors in that get sent to zero by the new matrix, $C_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The solution associated with this vector is $(e^{-x})\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = e^{-x}\begin{bmatrix} 0 \\ 1 \\ 2x \end{bmatrix}$. We stop after the second term because the vectors get sent to zero by $(A - \lambda I_3)^2$ and so will also get sent to zero by any higher power. Finally, we have to find the third solution.

$$(A - \lambda I_3)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can again pick any vector that is linearly independent from the first two; it seems natural to pick $\begin{bmatrix} 0 \\ 0 \\ 1. \end{bmatrix}$.
We must add the terms from the first 3 parts of the sum in order to calculate the solution for this vector.
$(e^{-x}) \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{x^2}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = e^{-x} \begin{bmatrix} x^2+1 \\ x \\ 3x \end{bmatrix}$
Therefore, the general solution is of the form
$\vec{y} = C_1 e^{-x} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + C_2 e^{-x} \begin{bmatrix} 0 \\ 1 \\ 2x \end{bmatrix} + C_3 e^{-x} \begin{bmatrix} x^2+1 \\ x \\ 3x \end{bmatrix}$