Math 32A Review Sheet

Tau Beta Pi - Boelter 6266

Contents

1 Parametric Equation

Parametric Equations: describe a particle's motion over time.
If a particle follows a 2-D curve described by C(t), its motion can also be described by the coordinates x(t) and y(t):

$$C(t) = (x(t), y(t))$$

When parametrizing any functino or curve, the goal is to isolate a variable and to express the isolated variable in terms of t.

1.1 Line

Line can be described as:

$$y = mx + b$$

where m stands for the slope and b represents a constant (the calue of y when x=0).
If a line passes through a point (a,b) with a slope of m its motion can be described as:

$$x(t) = a + rt;\ \ y(t) = b + st$$

such that m=$\frac{s}{r}$, $r \neq 0$
r and s represent how much x and y vary with t; and a and b represent the value of x when y is 0, (x(0)) and the value of y when x is 0, (y(0)).
If a line passes through two points M(a,b) and N(c,d) with unknow slope, the slope can be found as:

$$m = \frac{s}{r} = \frac{b-d}{c-a}$$

Therefore, the descriptions turn to:

$$x(t) = a + (c-a)t$$

$$y(t) = b + (b-d)t$$

1.2 Circle

A circle can be described as:

$$x(t) = a + Rcos(t);\ \ y(t) = b + Rsin(t)$$

where R is the radiuse of the circle.

1.3 Ellipse

Similar to a circle, an ellipse can be expressed as:

$$x(t) = a + Ccos(t);\ \ y(t) = b + Dsin(t)$$

where a and b reoresent the x and y coordinates of the center of the ellipse, and C and D represent the horizontal and verticle distance from the center to the edge repectively.

1.4 Cycloid

A cycloid is formed by the motion of a point on a circle as the circle rolls without slipping.

$$x(t) = t(t - sin(t));\ \ y(t) = (1-)cos(t))$$

2 Vectors in the Plane

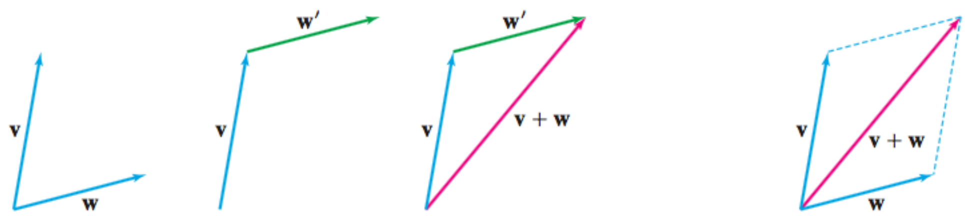
2.1 Vectors in the Plane

Two dimensional vector v is determined by two points in a plane (an initial point + a erminal point):

$$v = \vec{PQ}$$

where P = (a1, b1) and Q = (a2, b2)
Length of magnitude: $||v||.Andcanbecalculatedby : ||v|| = ||PQ|| = \sqrt{a^2 + b^2}$.
Parallal: The lines through v and w are parallel.
Translation: When a vector is moved to begin at a new point without changing its length or direction. (If a vector is the translation of another vector, aka. have the same components, these two vectors are defined as equivalent to each other).

2.2 Vector Algebra



(A) The vector sum **v** + **w** (B) Addition via the Parallelogram Law

Basic Properties of Vector Algebra For all vectors u, v, w and for all scalars λ
Commutative Law: v + w = w + v
Associative Law: u + (v + w) = (u + v) + w
Distributive Law for Scalars: $\lambda(v + w) = \lambda v + \lambda w$

Linear Combination: Every vector is a linear combinatio of other vectors.
For every vector $\vec{u}, \vec{u} = \lambda\vec{v} + \mu\vec{w}$, forming a parallelogram.
Unit Vector: $e_v = \frac{1}{||v||}v$. $||e_v|| = 1$.
Triangle Inequality: $|| + w|| \geq ||v|| + ||w||$.

2.3 Vectors in 3D

Right Hand Rule: Sphere of radius R and center (a, b, c):

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2$$

Cylinder of radius R with vertical axis through (a, b, 0):

$$(x - a)^2 + (y - b)^2 = R^2$$

Distance Formula:

$$P = (a1, b1, c1) and Q = (a2, b2, c2)$$
$$||P - Q|| = ||v|| = ||PQ|| = \sqrt{(a2 - a1)^2 + (b2 - b1)^2 + (c2 - c1)^2}$$

Vector Parametrization:
Equations for the line through $P_0 = (x_0, y_0, z_0)$ with direction vector v = (a, b, c):
Vectorparametrization: $r(t) = (x_0, y_0, z_0) + t(a, b, c)$
Parametric equations: $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$

2.4 Dot Product

The dot product of v = (a1,b1,c1) and w = (a2,b2,c2) is

$$v \cdot w = a1a2 + b1b2 + c1c2$$

Basic Properties:
- Commutativity:$v \cdot w = w \cdot v$
- Pullingoutscalars: $(\lambda v) \cdot w = v \cdot (\lambda w) = \lambda(v \cdot w)$
- DistributiveLaw: $u \cdot (v + w) = u \cdot v + u \cdot w$
  $(v + w) \cdot u = v \cdot u + w \cdot u$
- $v \cdot v = ||v||^2$ - $v \cdot w = ||v|| \cdot ||w||cos(\theta)$, where θ is the angle between v and w

2.4.1 Angles Between Lines

-Perpendicular: $v \cdot w$=0.
-Acute: if $v \cdot w > 0$.
-obtuse if $v \cdot w < 0$.
-Every vector u has a decomposition $u = u_{||v} + u_{\perp v}$, where $u_{||v}$ is parallel to v, and $u_{\perp v}$ is orthogonal to v. The vector $u_{||v}$ is called the projection of u along v.
-Let $e_v = \frac{v}{||v||}$, Then

$$u_{||v} = (\frac{u \cdot v}{v \cdot v})v = (\frac{u \cdot v}{||v||^2})v = (\frac{u \cdot v}{||v||})e_v$$

-The coefficient $\frac{u \cdot v}{||v||}$ is called the component of u along v.

2.5 Cross Product

Determinants:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cross Product:
The coross Product between $v =< v_1, v_2, v_3 > andw =< w_1, w_2, w_3 >$ is the symbolic determinant:

$$v \times w = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} i - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} j + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} k$$

Basic Properties:
- v×w is orthogonal to v and w.
- v×w has length $|v| \cdot |w| \cdot sin\theta.(\theta istheanglebetweenvandw, 0 \leq \theta \leq \pi)$.
- v, w, v × w is a right-handed system.
- w×v=-v×w
- v×w=0 iff w = λv for some scalar or v=0.
- (λv)×w=v×(λw)=λ(v×w)
- (u+v)×w=u×w+v×w v × (u + w) = v × u + v × w
- For standard basis vectors: i×j=k, j×k=i, k×i=j

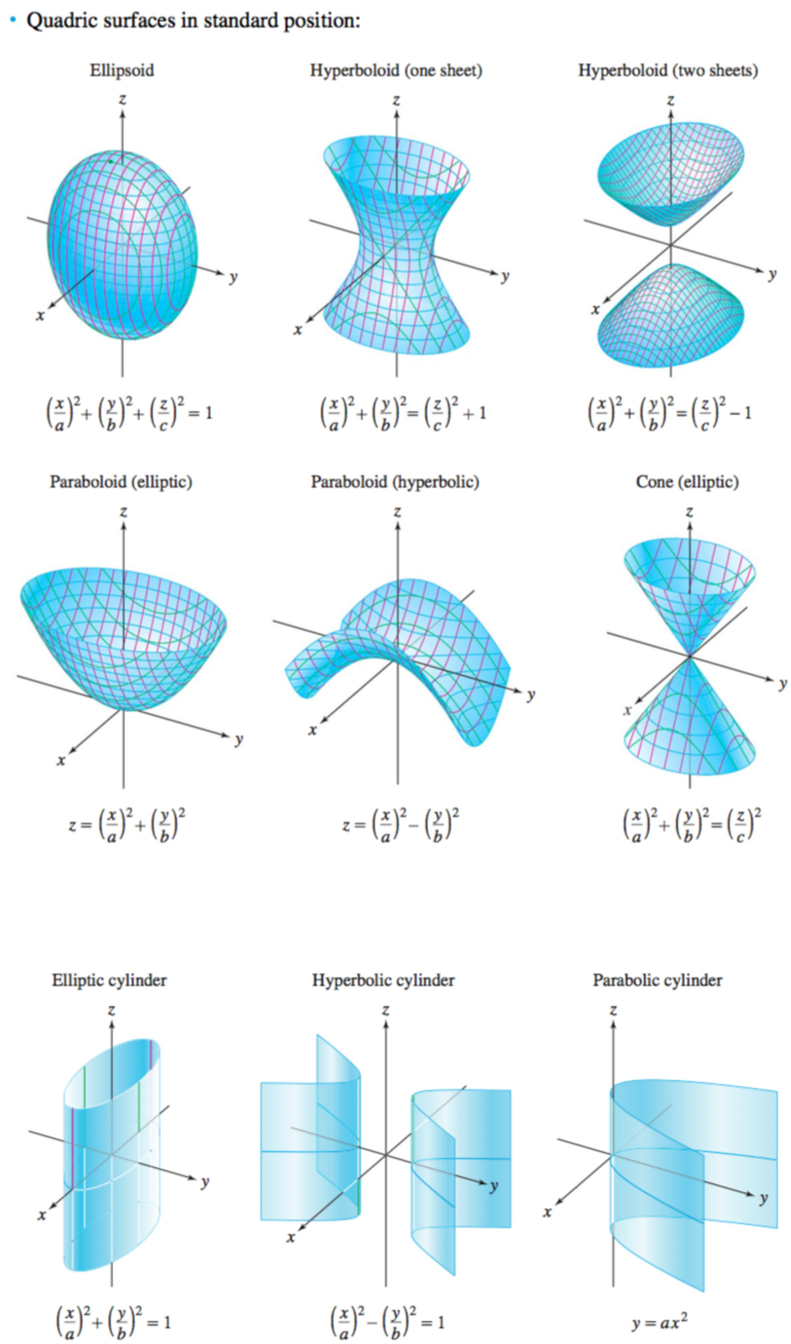
Geometries:
- Parallelogram spanned by v and w has area: $||v \times w||$
- Triangle spanned by v and w has area: $\frac{||v \times w||}{2}$
- Parallelepiped spanned by u, v, and w has volume: $|u \cdot (v \times w)|$

2.6 Planes in 3D

- Equation of plane through $P_0 = (x_0, y_0, z_0)$ with normal vector n = (a, b, c):
Vector form: n · (x, y, z) = d
Scalar forms: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad ax + by + cz = d$
where $d = n \cdot (x_0, y_0, z_0) = ax_0 + by_0 + cz_0$.
- For a plane through three points P , Q, R that are not collinear:

$$n = \vec{PQ} \times \vec{PR}, \qquad d = n \cdot (x_0, y_0, z_0), where P = (x_0, y_0, z_0)$$

2.7 Quadratic Surfaces



3 Calculus of Vector-Valued Functions

Vector-valued function: a function of the form

$$r(t) = (x(t), y(t), z(t)) = x(t)i + y(t)j + z(t)k$$

3.1 Calculation Rules

• Differentiation rules:
– Sum Rule: $(r_1(t) + r_2(t))' = r_1'(t) + r_2'(t)$
– Constant Multiple Rule: $(cr(t))' = cr'(t)$
– Chain Rule: $\frac{d}{dt}r(g(t)) = g'(t)r'(g(t))$

• Product Rules:
Scalar times vector: $\frac{d}{dt} = f'(t)r(t) + f(t)r'(t)$
Dot product: $\frac{d}{dt} = r_1(t) \cdot r_2(t) = r_1'(t) \cdot r_2(t) + r_1(t) \cdot r_2'(t)$
Cross product: $\frac{d}{dt} = r_1(t) \times r_2(t) = r_1'(t) \times r_2(t) + r_1(t) \times r_2'(t)$

• The tangent vector or velocity vector: derivative $r'(t_0)$.
• The Fundamental Theorem for vector-valued functions: If r(t) is continuous and R(t) is an antiderivative of r(t), then:

$$\int_a^b r(t)dt = R(b) - R(a)$$

3.2 Arc Length, Speed, and Curvature

• Arc length function: $s(t) = \int_a^b ||r'(u)||du$
• Speed: $v(t) = \frac{ds}{dt} = \int_a^b ||r'(t)||dt$
• r(s) is an arc length parametrization if $||r'(s)|| = 1$ for all s. In this case, the length of the path for $a \leq s \leq b$ is b - a.
• Regular parametrization r(t): $r'(t) \neq 0$ for all t. The unit tangent vector for regular r(t): $T(t) = \frac{r'(t)}{||r'(t)||}$
• Curvature: $k(s) = \frac{dT}{ds}$, where r(s) is an arc length parametrization or $k(s) = \frac{1}{v(t)}||\frac{dT}{dt}||$ if r(t) is not an arc length parametrization.
• Formula valid for arbitrary regular parametrizations:

$$k(t) = \frac{||r'(t) \times r''(t)||}{||r''(t)||^3}$$

• The curvature at a point on a graph y = f (x) in the plane:

$$k(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}$$

• Unit normal vector $N(t) = \frac{T'(t)}{||T'(t)||}$.
• $T'(t) = k(t)v(t)N(t)$ • The binormal vector: $B = T \times N$.

3.3 Motion in 3D

For an object whose path is described by a vector-valued function r(t),

$$v(t) = r'(t), v(t) = ||v(t)||, a(t) = r''(t)$$

The acceleration vector a is the sum of a tangential component (reflecting change in speed) and a normal component (reflecting change in direction):

$$a(t) = a_T(t)T(t) + a_N(t)N(t)$$

- Unit tangent vector: $T(t) = \frac{v(t)}{||v(t)||}$ - Unit normal vector: $N(t) = \frac{T'(t)}{||T'(t)||}$ - Tangential component:

$$a_T = v'(t) = a \cdot T = \frac{a \cdot v}{||v||}$$

$$a_T T = (\frac{a \cdot v}{v \cdot v})v$$

- Normal component:

$$a_N = k(t)v(t)^2 = \sqrt{||a||^2 - |a_T|^2}$$

$$a_N N = a - a_T T = a - (\frac{a \cdot v}{v \cdot v})v$$

4 Differentiation in Several Variables

4.1 Limits and Continuity

• The limit of a product $f(x, y) = h(x)g(y)$ is a product of limits:

$$\lim_{(x,y)\to(a,b)} f(x, y) = (\lim_{x\to a} h(x))(\lim_{y\to b} g(y))$$

• A function f of two variables is continuous at P = (a, b) if:

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

• To prove that a limit does not exist, it is enough to show that the limits obtained along two different paths are not equal.

4.2 Partial Derivatives

• The partial derivatives of f (x, y) are defined as the limits

$$f_x(a, b) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

(Compute fx by holding y constant, and compute fy by holding x constant.)
• For small changes $\Delta x and \Delta y$,

$$f(a + \Delta x, b) - f(a, b) = f_x(a, b)\Delta x$$

$$f(a, b + \Delta y) - f(a, b) = f_y(a, b)\Delta y$$

• The second-order partial derivatives:

$$\frac{\partial^2}{\partial x^2} f = f_{xx}, \quad \frac{\partial^2}{\partial y \partial x} f = f_{xy}, \quad \frac{\partial^2}{\partial x \partial y} f = f_{yx}, \quad \frac{\partial^2}{\partial y^2} f = f_{yy}$$

• Clairaut's Theorem: mixed partials are equal (fxy = fyx)provided that fxy and fyx are continuous.
• More generally, higher order partial derivatives may be computed in any order. For example, fxyyz = fyxzy if f is a function of x, y, z whose fourth-order partial derivatives are continuous.

4.3 Gradient of a Function

• The gradient of a function f is the vector of partial derivatives:

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \;\; or \;\; \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

• Chain Rule for Paths:

$$\frac{d}{dt} f(r(t)) = \nabla f_{r(t)} \cdot r'(t)$$

• Basic geometric properties of the gradient (assume $\nabla f_P \neq 0$): – ∇f_P points in the direction of maximum rate of increase. The maximum rate of increase is $||\nabla f_P||$.
– $-\nabla f_P$ points in the direction of maximum rate of decrease. The maximum rate of decrease is $-||\nabla f_P||$.
– ∇f_P is orthogonal to the level curve (or surface) through P.
• Equation of the tangent plane to the level surface F (x, y, z) = k at P = (a, b, c):

$$\nabla f_P \cdot (x - a, y - b, z - c) = 0$$

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

4.4 Optimization in Several Variables

• P = (a,b) is a critical point of f(x,y) if
– $f_x(a, b) = 0 or f_x(a, b)$ does not exist, and
– $f_y(a, b) = 0 or f_y(a, b)$ does not exist. • The local minimum or maximum values of f occur at critical points.
• The discriminant of f (x, y) at P = (a, b) is the quantity

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b)$$

• Second Derivative Test: If P = (a, b) is a critical point of f (x, y), then

$$D(a, b) > 0, \quad f_{xx}(a, b) > 0 \quad \Rightarrow \quad f(a, b) \textit{ is a local minimum}$$

$$D(a, b) > 0, \quad f_{xx}(a, b) < 0 \quad \Rightarrow f(a, b) \textit{ is a local maximum}$$

$$D(a, b) < 0 \quad \Rightarrow \quad \textit{saddle point}$$

$$D(a, b) = 0 \quad \Rightarrow \quad \textit{test inconclusive}$$

• A point P is an interior point of a domain D if D contains some open disk D(P,r) centered at P. A point P is a boundary point of D if every open disk D(P,r) contains points in D and points not in D. The interior of D is the set of all interior points, and the boundary is the set of all boundary points. A domain is closed if it contains all of its boundary points and open if it is equal to its interior.
• Existence and location of global extrema: If f is continuous and D is closed and bounded, then
– f takes on both a minimum and a maximum value on D.
– The extreme values occur either at critical points in the interior of D or at points on the boundary of D.