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1 Parametric Equation

Parametric Equations: describe a particle’s motion over time. If a particle follows a 2-D curve described by \( C(t) \), its motion can also be described by the coordinates \( x(t) \) and \( y(t) \):

\[
C(t) = (x(t), y(t))
\]

When parametrizing any function or curve, the goal is to isolate a variable and to express the isolated variable in terms of \( t \).

1.1 Line

Line can be described as:

\[
y = mx + b
\]

where \( m \) stands for the slope and \( b \) represents a constant (the value of \( y \) when \( x = 0 \)).

If a line passes through a point \((a,b)\) with a slope of \( m \) its motion can be described as:

\[
x(t) = a + rt; \quad y(t) = b + st
\]

such that \( m = \frac{s}{r} \), \( r \neq 0 \)

\( r \) and \( s \) represent how much \( x \) and \( y \) vary with \( t \); and \( a \) and \( b \) represent the value of \( x \) when \( y = 0 \), \((x(0))\) and the value of \( y \) when \( x = 0 \), \((y(0))\).

If a line passes through two points \( M(a,b) \) and \( N(c,d) \) with unknown slope, the slope can be found as:

\[
m = s = \frac{b - d}{c - a}
\]

Therefore, the descriptions turn to:

\[
x(t) = a + (c - a)t
\]
\[
y(t) = b + (b - d)t
\]

1.2 Circle

A circle can be described as:

\[
x(t) = a + R\cos(t); \quad y(t) = b + R\sin(t)
\]

where \( R \) is the radius of the circle.

1.3 Ellipse

Similar to a circle, an ellipse can be expressed as:

\[
x(t) = a + C\cos(t); \quad y(t) = b + D\sin(t)
\]

where \( a \) and \( b \) represent the \( x \) and \( y \) coordinates of the center of the ellipse, and \( C \) and \( D \) represent the horizontal and vertical distance from the center to the edge respectively.

1.4 Cycloid

A cycloid is formed by the motion of a point on a circle as the circle rolls without slipping.

\[
x(t) = t(t - \sin(t)); \quad y(t) = (1 - \cos(t))
\]
2 Vectors in the Plane

2.1 Vectors in the Plane

Two dimensional vector $v$ is determined by two points in a plane (an initial point + a terminal point):

$$ v = \overrightarrow{PQ} $$

where $P = (a_1, b_1)$ and $Q = (a_2, b_2)$

Length of magnitude: $||v||$. And can be calculated by:

$$ ||v|| = ||\overrightarrow{PQ}|| = \sqrt{a^2 + b^2}. $$

Parallel: The lines through $v$ and $w$ are parallel.

Translation: When a vector is moved to begin at a new point without changing its length or direction. (If a vector is the translation of another vector, aka. have the same components, these two vectors are defined as equivalent to each other).

2.2 Vector Algebra

Basic Properties of Vector Algebra

For all vectors $u$, $v$, $w$ and for all scalars $\lambda$

Commutative Law: $v + w = w + v$

Associative Law: $u + (v + w) = (u + v) + w$

Distributive Law for Scalars: $\lambda(v + w) = \lambda v + \lambda w$

Linear Combination: Every vector is a linear combination of other vectors.

For every vector $\vec{u}, \vec{u} = \lambda \vec{v} + \mu \vec{w}$, forming a parallelogram.

Unit Vector: $e_v = \frac{1}{||v||} v$. $||e_v|| = 1$.

Triangle Inequality: $||v + w|| \geq ||v|| + ||w||$.

2.3 Vectors in 3D

Right Hand Rule: Sphere of radius $R$ and center $(a, b, c)$:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2$$

Cylinder of radius $R$ with vertical axis through $(a, b, 0)$:

$$(x - a)^2 + (y - b)^2 = R^2$$

Distance Formula:

$$ ||P - Q|| = ||v|| = ||\overrightarrow{PQ}|| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2} $$

Vector Parametrization:

Equations for the line through $P_0 = (x_0, y_0, z_0)$ with direction vector $v = (a, b, c)$:

Vector parametrization: $r(t) = (x_0, y_0, z_0) + t(a, b, c)$

Parametric equations: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$
2.4 Dot Product

The dot product of \( v = (a_1, b_1, c_1) \) and \( w = (a_2, b_2, c_2) \) is

\[
v \cdot w = a_1 a_2 + b_1 b_2 + c_1 c_2
\]

Basic Properties:
- Commutativity: \( v \cdot w = w \cdot v \)
- Pulling out scalars: \( (\lambda v) \cdot w = v \cdot (\lambda w) = \lambda (v \cdot w) \)
- Distributive Law: \( u \cdot (v + w) = u \cdot v + u \cdot w \)
- \( v \cdot v = ||v||^2 \) - \( v \cdot w = ||v|| \cdot ||w|| \cos(\theta) \), where \( \theta \) is the angle between \( v \) and \( w \)

2.4.1 Angles Between Lines

- Perpendicular: \( v \cdot w = 0 \).
- Acute: if \( v \cdot w > 0 \).
- Obtuse if \( v \cdot w < 0 \).
- Every vector \( u \) has a decomposition \( u = u_{||v} + u_{\perp v} \), where \( u_{||v} \) is parallel to \( v \), and \( u_{\perp v} \) is orthogonal to \( v \).

Let \( e_v = \frac{v}{||v||} \), Then

\[
u_{||v} = \left( \frac{u \cdot v}{v \cdot v} \right) v = \left( \frac{u \cdot v}{||v||} \right) e_v
\]

The coefficient \( \frac{u \cdot v}{||v||} \) is called the component of \( u \) along \( v \).

2.5 Cross Product

Determinants:

\[
\begin{vmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}
\]

\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\]

Cross Product:

The cross product between \( v = <v_1, v_2, v_3> \) and \( w = <w_1, w_2, w_3> \) is the symbolic determinant:

\[
v \times w = \begin{bmatrix} v_2 & v_3 \\ -v_3 & v_1 \end{bmatrix} i - \begin{bmatrix} v_1 & v_3 \\ -v_3 & v_2 \end{bmatrix} j + \begin{bmatrix} v_1 & v_2 \\ -v_2 & v_1 \end{bmatrix} k
\]

Basic Properties:
- \( v \times w \) is orthogonal to \( v \) and \( w \).
- \( v \times w \) has length \( ||v|| \cdot ||w|| \cdot \sin(\theta) \) where \( \theta \) is the angle between \( v \) and \( w \), \( 0 \leq \theta \leq \pi \).
- \( v, w, v \times w \) is a right-handed system.
- \( w \times v = -v \times w \)
- \( v \times w = 0 \) iff \( \lambda v \) for some scalar \( v = 0 \).
- \( (\lambda v) \times w = v \times (\lambda w) = \lambda (v \times w) \)
- \( (u + v) \times w = u \times w + v \times w \)
- For standard basis vectors: \( i \times j = k, k \times i = j \)

Geometries:
- Parallelogram spanned by \( v \) and \( w \) has area: \( ||v \times w|| \)
- Triangle spanned by \( v \) and \( w \) has area: \( \frac{||v \times w||}{2} \)
- Parallelepiped spanned by \( u, v, w \) has volume: \( |u \cdot (v \times w)| \)
2.6 Planes in 3D

- Equation of plane through $P_0 = (x_0, y_0, z_0)$ with normal vector $n = (a, b, c)$:
  Vector form: $n \cdot (x, y, z) = d$
  Scalar forms: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad ax + by + cz = d$
  where $d = n \cdot (x_0, y_0, z_0) = ax_0 + by_0 + cz_0$.

- For a plane through three points $P$, $Q$, $R$ that are not collinear:
  \[ n = \vec{PQ} \times \vec{PR}, \quad d = n \cdot (x_0, y_0, z_0), \text{where} \ P = (x_0, y_0, z_0) \]

2.7 Quadratic Surfaces
3 Calculus of Vector-Valued Functions

Vector-valued function: a function of the form
\[ r(t) = (x(t), y(t), z(t)) = x(t)i + y(t)j + z(t)k \]

3.1 Calculation Rules

- **Differentiation rules:**
  - Sum Rule: \( (r_1(t) + r_2(t))' = r_1'(t) + r_2'(t) \)
  - Constant Multiple Rule: \( (cr(t))' = cr'(t) \)
  - Chain Rule: \( \frac{d}{dt}r(g(t)) = g'(t)r'(g(t)) \)

- **Product Rules:**

Scalar times vector: \( \frac{d}{dt} = f'(t)r(t) + f(t)r'(t) \)

Dot product: \( \frac{d}{dt} = r_1(t) \cdot r_2(t) = r_1'(t) \cdot r_2(t) + r_1(t) \cdot r_2'(t) \)

Cross product: \( \frac{d}{dt} = r_1(t) \times r_2(t) = r_1'(t) \times r_2(t) + r_1(t) \times r_2'(t) \)

- The tangent vector or velocity vector: derivative \( r'(t_0) \).
- The Fundamental Theorem for vector-valued functions: If \( r(t) \) is continuous and \( R(t) \) is an antiderivative of \( r(t) \), then:
  \[ \int_a^b r(t)dt = R(b) - R(a) \]

3.2 Arc Length, Speed, and Curvature

- **Arc length function:** \( s(t) = \int_a^b ||r'(u)||du \)
- **Speed:** \( v(t) = \frac{ds}{dt} = \int_a^b ||r'(t)||dt \)
  \( r(s) \) is an arc length parametrization if \( ||r'(s)|| = 1 \) for all \( s \). In this case, the length of the path for \( a \leq s \leq b \) is \( b - a \).
  - **Regular parametrization** \( r(t) \); \( r'(t) \neq 0 \) for all \( t \). The unit tangent vector for regular \( r(t) \):
    \[ T(t) = \frac{r'(t)}{||r'(t)||} \]
  - **Curvature:** \( k(s) = \frac{dT}{ds} \), where \( r(s) \) is an arc length parametrization or \( k(s) = \frac{1}{v(t)} \frac{dT}{dt} \) if \( r(t) \) is not an arc length parametrization.
  - Formula valid for arbitrary regular parametrizations:
    \[ k(t) = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3} \]
  - The curvature at a point on a graph \( y = f(x) \) in the plane:
    \[ k(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} \]
  - Unit normal vector \( N(t) = \frac{T'(t)}{||T'(t)||} \).
  - \( T'(t) = k(t)v(t)N(t) \)
  - The binormal vector: \( B = T \times N \).

3.3 Motion in 3D

For an object whose path is described by a vector-valued function \( r(t) \),
\[ v(t) = r'(t), v(t) = ||v(t)||, a(t) = r''(t) \]

The acceleration vector \( a \) is the sum of a tangential component (reflecting change in speed) and a normal component (reflecting change in direction):
\[ a(t) = a_T(t)T(t) + a_N(t)N(t) \]
- Unit tangent vector: \( T(t) = \frac{v(t)}{||v(t)||} \) - Unit normal vector: \( N(t) = \frac{T'(t)}{||T'(t)||} \) - Tangential component:

\[
a_T = v'(t) = a \cdot T = \frac{a \cdot v}{||v||}
\]

\[
a_T T = \left(\frac{a \cdot v}{v \cdot v}\right) v
\]

- Normal component:

\[
a_N = k(t)v(t)^2 = \sqrt{||a||^2 - |a_T|^2}
\]

\[
a_N N = a - a_T T = a - \left(\frac{a \cdot v}{v \cdot v}\right) v
\]

4 Differentiation in Several Variables

4.1 Limits and Continuity

- The limit of a product \( f(x, y) = h(x)g(y) \) is a product of limits:

\[
\lim_{(x,y) \to (a,b)} f(x, y) = (\lim_{x \to a} h(x))(\lim_{y \to b} g(y))
\]

- A function \( f \) of two variables is continuous at \( P = (a, b) \) if:

\[
\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)
\]

- To prove that a limit does not exist, it is enough to show that the limits obtained along two different paths are not equal.

4.2 Partial Derivatives

- The partial derivatives of \( f(x, y) \) are defined as the limits

\[
f_x(a, b) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}
\]

(Compute \( f_x \) by holding \( y \) constant, and compute \( f_y \) by holding \( x \) constant.)

- For small changes \( \Delta x \) and \( \Delta y \),

\[
f(a + \Delta x, b) - f(a, b) = f_x(a, b) \Delta x
\]

\[
f(a, b + \Delta y) - f(a, b) = f_y(a, b) \Delta y
\]

- The second-order partial derivatives:

\[
\frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 f}{\partial y \partial x} = f_{xy}, \quad \frac{\partial^2 f}{\partial x \partial y} = f_{yx}, \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}
\]

- Clairaut’s Theorem: mixed partials are equal \( (f_{xy} = f_{yx}) \) provided that \( f_{xy} \) and \( f_{yx} \) are continuous.

- More generally, higher order partial derivatives may be computed in any order. For example, \( f_{xyyz} = f_{yxzy} \) if \( f \) is a function of \( x, y, z \) whose fourth-order partial derivatives are continuous.
4.3 Gradient of a Function

- The gradient of a function $f$ is the vector of partial derivatives:

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \quad \text{or} \quad \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

- Chain Rule for Paths:

$$\frac{df}{dt}(r(t)) = \nabla f_{r(t)} \cdot r'(t)$$

- Basic geometric properties of the gradient (assume $\nabla f \neq 0$):
  - $\nabla f$ points in the direction of maximum rate of increase. The maximum rate of increase is $||\nabla f||$.
  - $-\nabla f$ points in the direction of maximum rate of decrease. The maximum rate of decrease is $-||\nabla f||$.
  - $\nabla f$ is orthogonal to the level curve (or surface) through $P$.

- Equation of the tangent plane to the level surface $F(x, y, z) = k$ at $P = (a, b, c)$:

$$\nabla f_P \cdot (x-a, y-b, z-c) = 0$$

$$F_x(a, b, c)(x-a) + F_y(a, b, c)(y-b) + F_z(a, b, c)(z-c) = 0$$

4.4 Optimization in Several Variables

- $P = (a,b)$ is a critical point of $f(x,y)$ if
  - $f_x(a,b) = 0$ or $f_x(a,b)$ does not exist, and
  - $f_y(a,b) = 0$ or $f_y(a,b)$ does not exist.
- The local minimum or maximum values of $f$ occur at critical points.

- The discriminant of $f(x, y)$ at $P = (a, b)$ is the quantity

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)$$

- Second Derivative Test: If $P = (a, b)$ is a critical point of $f(x, y)$, then

$$D(a,b) > 0, \quad f_{xx}(a,b) > 0 \quad \Rightarrow \quad f(a,b) \text{ is a local minimum}$$

$$D(a,b) > 0, \quad f_{xx}(a,b) < 0 \quad \Rightarrow \quad f(a,b) \text{ is a local maximum}$$

$$D(a,b) < 0 \quad \Rightarrow \quad \text{saddle point}$$

$$D(a,b) = 0 \quad \Rightarrow \quad \text{test inconclusive}$$

- A point $P$ is an interior point of a domain $D$ if $D$ contains some open disk $D(P,r)$ centered at $P$. A point $P$ is a boundary point of $D$ if every open disk $D(P,r)$ contains points in $D$ and points not in $D$. The interior of $D$ is the set of all interior points, and the boundary is the set of all boundary points. A domain is closed if it contains all of its boundary points and open if it is equal to its interior.

- Existence and location of global extrema: If $f$ is continuous and $D$ is closed and bounded, then
  - $f$ takes on both a minimum and a maximum value on $D$.
  - The extreme values occur either at critical points in the interior of $D$ or at points on the boundary of $D$. 

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